Bi-Linear Flocks and Translation Planes

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Flocks of Arbitrary Cones

A **flock** of a cone in $\text{PG}(3,q)$ is a set of $q$ planes which partition the points of the cone (except for the vertex).

The **carrier** (or base) of the cone may be any set of points in the plane $x_3 = 0$, which can always be taken as one of the planes in the flock.
Linear Flocks
The Flock Condition

A set of \( q \) planes (indexed by elements of \( \text{GF}(q) \)) not passing through \( V(0,0,0,1) \) can be described by three functions \( f,g,h: \text{GF}(q) \rightarrow \text{GF}(q) \) defined by the equations of the planes;

\[
\pi_t : f(t)x_0 + g(t)x_1 + h(t)x_2 - x_3 = 0.
\]

Such a set of \( q \) planes is a flock of the cone with carrier \( C \) if and only if

\[
t \rightarrow f(t)a + g(t)b + h(t)c
\]

is a permutation of \( \text{GF}(q) \) for each \( (a,b,c,0) \in C \).

Denote a flock by \( \mathcal{F} = \mathcal{F}(f,g,h) \).
Translation Planes and Spread Sets

A *translation plane* is a projective plane whose translation group acts regularly on its affine points.

A (matrix) *spread set* on GF(q)^2 is a set of q^2 2x2 matrices, including the zero matrix, such that the difference of any two distinct matrices is invertible.

By a standard construction, every spread set gives rise to a translation plane. Any translation plane will have many spread set representations.
Flokki and Spread Sets

The cone over carrier $xz^\alpha = y^{\alpha+1}$ (with $x \rightarrow x^\alpha$ an automorphism) has a flock $\mathcal{F} = \mathcal{F}(-f^\alpha, g^\alpha, t)$ iff

$-x^{\alpha+1} f(t)^\alpha + x g(t)^\alpha + t$ is a permutation $\forall x$.

Using $\Delta f = f(t_1) - f(t_2)$, $\Delta g = g(t_1) - g(t_2)$ and $\Delta t = t_1 - t_2$ we have that

$-x^\alpha(\Delta f)^\alpha + (\Delta g)^\alpha + (\Delta t)/x \neq 0$ if $t_1 \neq t_2$.

With $u^\alpha = (\Delta t)/x$, this is equivalent to

$u^\alpha(u + \Delta g) - \Delta t \Delta f \neq 0$ if $t_1 \neq t_2$.

$$\begin{pmatrix} u + g(t) & f(t) \\ t & u^\alpha \end{pmatrix}$$

is a spread set.
A special case

Consider the special case of flokki in PG(3,q^2) where we take α = q. A flock of the form \( \mathcal{F}(-f^q,0,t) \) gives rise to the spread set:

\[
\begin{pmatrix}
u & f(t) \\
t & u^q \\
t & u^q
\end{pmatrix}, \quad u, t \in GF(q^2).
\]

There is a technique, called “algebraic lifting” which will take \textit{any} spread over GF(q) and produce one of the above form.
Star Flocks

A flock in which all the planes meet at a unique point is called a (proper) **star flock**. The intersection point is called the **star point**.

A flock \( \mathcal{F}(f,g,h) \) in which one of the functions is the zero function is a star flock.

Star flocks are most easily studied by considering the space dual, where points and planes are interchanged and lines get mapped to lines. In this setting, a flock consists of \( q \) points in the plane corresponding to the star point, none of which is in the plane corresponding to the vertex of the cone.
Dual Setting for Star Flocks

Vertex plane →

Line joining two flock points does not meet any line which is the image of a generator of the cone.

← Star point plane
Rédei Blocking Sets

A **blocking set** is a set of points in a plane which every line of the plane intersects.

A **Rédei blocking set** is a blocking set consisting of $q+N$ points in a plane of order $q$ containing a largest collinear set of $N$ points (called a *Rédei line*).

Let a Rédei line be the line at infinity of the plane. The set $\mathcal{V}$ of the $q$ affine points of this blocking set has the property that the join of any of its points meets the line at infinity in a blocking set point.
The “Big” Theorem

**Theorem:** (Ball, 2002) Let $f$ be a function from $\text{GF}(q)$ to $\text{GF}(q)$, $q = p^h$ for some prime $p$, and let $N$ be the number of directions determined by $f$. Let $s = p^e$ be maximal such that any line with a direction determined by $f$ that is incident with a point of $\mathcal{U}$ is incident with a multiple of $s$ points of $\mathcal{U}$. One of the following holds:

(i) $s = 1$ and $(q+3)/2 \leq N \leq q+1$;
(ii) $\text{GF}(s)$ a subfield of $\text{GF}(q)$ & $q/s + 1 \leq N \leq (q-1)/(s-1)$;
(iii) $s = q$ and $N = 1$.

Moreover, if $s > 2$ then $f$ is an $s$-linearized polynomial.
Putting it all together

A star flock \( \mathcal{F}(-f^q,0,t) \) of the cone over the carrier \( xz^q = y^{q+1} \) in PG(3,\(q^2\)) when viewed in the dual setting form the affine part of a Rédei blocking set having \( N \leq q^2 - q \) directions (slopes).

Note: The relationship between star flocks and Rédei blocking sets is general, it is only the bound on \( N \) which follows from this particular choice of cone.
Small Cases

$q = 2$: We would have $N \leq 2$, but by the “Big” Theorem, this implies $N = 1$. So, the star flock is a linear flock and the translation plane is the Desarguesian plane.

$q = 3$: We have $N \leq 6$ which implies $N = 1$, 4 or 6. As above $N = 1$ gives the linear flock and the Desarguesian plane. $N = 4$ gives an affine subplane of order 3. Since the function involved is 3-linearized, the translation plane is a semifield plane, but the only semifield plane of order 9 is the Desarguesian plane. This leaves the case $N = 6 (= q^2 + 3/2)$. 
Bi-Linear Flocks
Bi-Linear Flocks

In PG(2,q), q odd, consider the q affine points \((x,f(x))\) where:

\[
f(x) = x^{\frac{q^2+1}{2}} = \begin{cases} 
  x & \text{if } x = \checkmark \\
  -x & \text{if } x = \times 
\end{cases}
\]

These points lie on two lines and the number of slopes determined by any two points of the set is \(q+3/2\). Together with the infinite points corresponding to these slopes, we have a Rédei blocking set with \(N = q+3/2\) known as a **projective triangle**.
... and Their Translation Planes

The spread sets obtained from these bi-linear flocks have:

\[ f(t) = \sqrt{\alpha(t^q)}^{\frac{q^2+1}{2}} \]

where \( \alpha \) is any non-zero element of GF(q) which is not a square in GF(q).

The translation planes are lifted regular nearfield planes. In the order 9 case, this is a Hall plane (which happens to also be a nearfield plane).