For any set $S$, an $r$-partition of $S$ is a set of $r$ nonempty disjoint sets whose union is $S$. The elements of the partition are called blocks of the partition.

Ex: A 2-partition of $[7]$ is given by ${\{1,3,4,5\}, \{2,6,7\}}$.

Partitions as distributions. Distinct objects into indistinct boxes.

A Stirling number of the second kind, $S(n,k)$, is the number of $k$-partitions of an $n$-set, i.e., the number of distributions of $n$ distinct objects into $k$ identical boxes such that each box receives at least one object.

Example: $S(4,1) = 1$ since the only partition with 1 block is $\{1234\}$.

$S(4,2) = 7$ since $\{1, 234\}, \{2, 134\}, \{3, 124\}, \{4, 123\}, \{12, 34\}, \{13, 24\}, \{14, 23\}$ are the only 2-partitions.

$S(4,3) = 6$ since $\{12, 3, 4\}, \{13, 2, 4\}, \{14, 2, 3\}, \{23, 1, 4\}, \{24, 1, 3\}, \{34, 1, 2\}$ and $S(4,4) = 1$ since $\{1,2,3,4\}$ is the only 4-partition.

We define $S(0,0) = 1$, but it is easy to show that

$S(n,1) = S(n,n) = 1$ for all $n$.

$S(n,2) = 2^{n-1} - 1$ and $S(n,n-1) = \binom{n}{2}$.

The Bell numbers, $B(n)$ count the total number of partitions of an $n$-set. Thus:

$$B(n) = \sum_{k=1}^{n} S(n,k), \quad \forall n \geq 1.$$

Ex: $B(4) = 1 + 7 + 6 + 1 = 15$.

We postpone the proofs of the formulae:

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-j)^n$$

and

$$B(n) = \frac{1}{e} \sum_{j=0}^{\infty} \frac{j^n}{j!}$$

A Pascal-like identity for Stirling numbers of the second kind:

$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k) \quad \text{for } n \geq 1, k \geq 1$$

Pf: Consider whether or not $n$ is in a block by itself.

Fill in third row of distribution table. (& finish top row)