The theory of design of experiments came into being largely through the work of R.A. Fisher and F. Yates in the early 1930's. They were motivated by questions of design of field experiments in agriculture. The applicability of this theory is now very widespread, much of the terminology still bears the stamp of its origins.
Consider an agricultural experiment. Suppose it is desired to compare the yield of $v$ different varieties of grain. It is quite possible that there would be an interaction between the environment (type of soil, rainfall, drainage, etc.) and the variety of grain which would alter the yields. So, $b$ blocks (sets of experimental plots) are chosen in which the environment is fairly consistent throughout the block. In other types of experiments, in which the environment might not be a factor, blocks could be distinguished as plots which receive a particular treatment (say, are given a particular type of fertilizer). In this way, the classification of the experimental plots into blocks and varieties can be used whenever there are two factors which may influence yield.
The obvious technique of growing every variety in a plot in every block, may, for large experiments be too costly or impractical. To deal with this, one would use smaller blocks which did not contain all of the varieties. Now the problem is one of comparison, to minimize the effects of chance due to incomplete blocks, we would want to design the blocks so that the probability of two varieties being compared (i.e. are in the same block) is the same for all pairs. This property would be called balance in the design. Statistical techniques, in particular Analysis of Variance (ANOVA), could then be used to reach conclusions about the experiment.
Designs

In modern usage we refer to a design as being an ordered pair \((X, \mathcal{B})\) where

- \(X\) is a set of elements called points, and
- \(\mathcal{B}\) is a collection (i.e., multiset) of subsets of \(X\) called blocks.

Note that in the literature when special types of designs were studied other terminology was traditionally used, and we will often use the more specialized versions to make reading the literature easier.
BIBD's

The most commonly studied type of design is the BIBD.

A BIBD is a set $X$ of $v \geq 2$ elements called *varieties* or *treatments* and a collection of $b > 0$ subsets of $X$, called *blocks*, such that the following conditions are satisfied:

- each block consists of exactly $k$ varieties, $v > k > 0$,
- each variety appears in exactly $r$ blocks, $r > 0$,
- each pair of varieties appear simultaneously in exactly $\lambda$ blocks, $\lambda > 0$.
BIBD's

The *incomplete* in the name of these designs refers to the condition $v > k$, i.e., the block size, $k$, is less than the total number of treatments, so no block contains all the varieties. If we allowed $v = k$, then all the conditions would be trivially satisfied and the resulting design would not be of much interest.

*Balanced* refers to the constancy of the $\lambda$ parameter.

If $\mathcal{B}$ is actually a set (when $\lambda > 1$) we refer to the design as a *simple* design, otherwise the design has *repeated blocks*.

BIBD's are often referred to as $(v,b,r,k,\lambda)$- designs.
An Example

An example of a (7,7,3,3,1)-design is given by the set $X$ consisting of the varieties, 1,2,3,4,5,6,7 and the following blocks:

$$\{1,2,4\} \ {2,3,5} \ {3,4,6} \ {4,5,7} \ {5,6,1} \ {6,7,2} \ {7,1,3}.$$  

It is easily seen that the block size $k$ is 3, and the repetition number $r$ is also 3. Each pair of varieties appears together in only one block, so $\lambda = 1$. 
Other Examples

A simple $(4,4,3,3,2)$-design is given by $X = \{1,2,3,4\}$ with blocks:

\[
\{1,2,3\} \quad \{2,3,4\} \quad \{3,4,1\} \quad \{4,1,2\}.
\]

A slightly larger example is the simple $(8,14,7,4,3)$-design on the set $X = \{1,2,3,4,5,6,7,8\}$ with blocks:

\[
\{1,3,7,8\} \quad \{1,2,4,8\} \quad \{2,3,5,8\} \quad \{3,4,6,8\} \quad \{4,5,7,8\} \quad \{1,5,6,8\} \\
\{2,6,7,8\} \quad \{1,2,3,6\} \quad \{1,2,5,7\} \quad \{1,3,4,5\} \quad \{1,4,6,7\} \quad \{2,3,4,7\} \\
\{2,4,5,6\} \quad \{3,5,6,7\}.
\]
Parameter Conditions

Some elementary counting arguments show that there are necessary conditions that the parameters of a BIBD must satisfy.

Theorem - Given a \((v,b,r,k,\lambda)\)-design,
- \(bk = vr\)
- \(r(k-1) = \lambda(v-1)\).

Proof: Consider the set of pairs \((x,B)\), where \(x\) is a variety and \(B\) is a block containing \(x\). By counting this set in two ways we arrive at the first equation. There are \(v\) possible values for \(x\), and since each appears in \(r\) blocks, \(vr\) will count the number of these pairs. On the other hand, there are \(b\) blocks and each contains \(k\) varieties, so \(bk\) also counts the number of these pairs.
Parameter Conditions

**Theorem** - Given a \((v,b,r,k,\lambda)\)-design,

- \(bk = vr\)
- \(r(k-1) = \lambda(v-1)\).

**Proof (cont.):** The second equation is also obtained by counting. Fix a particular variety, say \(p\), and count the number of pairs of varieties \{\(p,y\)\} where \(p\) and \(y\) appear in some block together and if the pair appears more than once it is multiply counted. There are \(v - 1\) possible choices of \(y\) and each such pair will appear in \(\lambda\) blocks together, so there are \(\lambda(v-1)\) such pairs. On the other hand \(p\) appears in \(r\) blocks and can be paired with \(k - 1\) other elements in such a block, thus \(r(k-1) = \lambda(v-1)\).
Parameter Conditions

The five parameters are not independent and we usually consider b and r as dependent ones and refer to \((v,k,\lambda)\) designs. Note that from the previous result we have:

\[ r = \frac{\lambda(v-1)}{k-1} \quad \text{and} \]
\[ b = \frac{vr}{k} = \frac{\lambda(v^2-v)}{k^2-k} \]

As \(b\) and \(r\) must be integers we obtain the necessary conditions:

**Corollary:** If a \((v,k,\lambda)\) design exists, then \(\lambda(v-1) \equiv 0 \ mod \ (k-1)\) and \(\lambda v(v-1) \equiv 0 \ mod \ k(k-1)\).
Incidence Matrix

Given a \((v,k,\lambda)\)-design, we can represent it with a \(v \times b\) matrix called the \textit{incidence matrix} of the design. The rows are labeled with the varieties of the design and the columns with the blocks. We put a 1 in the \((i,j)\)-th cell of the matrix if variety \(i\) is contained in block \(j\) and 0 otherwise.

Each row of the incidence matrix has \(r\) 1's, each column has \(k\) 1's and each pair of distinct rows has \(\lambda\) common 1's. These observations lead to a useful matrix identity.
Example

For the design \{1,2,3\} \ {2,3,4\} \ {3,4,1\} \ {4,1,2\} we construct the incidence matrix.

\[
\begin{array}{cccc}
\{1,2,3\} & \{2,3,4\} & \{3,4,1\} & \{4,1,2\} \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 1 & 0 & 1 \\
3 & 1 & 1 & 1 & 0 \\
4 & 0 & 1 & 1 & 1 \\
\end{array}
\]
Matrix Identity for Designs

**Theorem** - $A$, a $\{0,1\} v \times b$ matrix, is the incidence matrix of a $(v,k,\lambda)$-design, if and only if
\[ AA^T = (r-\lambda)I + \lambda J \]
and $u_v A = ku_b$. Where $I$ is the $v \times v$ identity matrix, $J$ is the $v \times v$ matrix of all 1's and $u_a$ is the all 1 vector of length $a$.

**Proof**: In the product, an off diagonal entry is the inner product of two distinct rows of $A$, which must be $\lambda$. A diagonal entry is the inner product of a row of $A$ with itself, and so equals $r$. The second condition just says that every column sum of $A$ is equal to $k$. 
Matrix Identity for Designs

**Theorem** - \( A \), a \( \{0,1\} \times b \) matrix, is the incidence matrix of a \((v,k,\lambda)\)-design, if and only if

\[
A A^T = (r-\lambda)I + \lambda J
\]

and \( u_v A = k u_b \). Where \( I \) is the \( v \times v \) identity matrix, \( J \) is the \( v \times v \) matrix of all 1's and \( u_a \) is the all 1 vector of length \( a \).

**Pf(cont)**: On the other hand, let \((X, \mathcal{B})\) be the design with incidence matrix \( A \). Clearly, \(|X| = v\), \(|\mathcal{B}| = b\) and every block has \( k \) elements follows from \( u_v A = k u_b \). From the expression for \( A A^T \) we get that every element is contained in \( r \) blocks and that every pair of distinct elements is contained in \( \lambda \) blocks. Thus, \((X, \mathcal{B})\) is a \((v,k,\lambda)\)-design.
Without the side condition of the last result we could not prove that the blocks had the same size. If we drop that condition, the same proof gives an equivalence with a more general type of design.

A pairwise balanced design (PBD) is a design \((X, \mathcal{B})\) where every pair of distinct points is contained in exactly \(\lambda\) blocks.

A PBD is regular if every point appears in exactly \(r\) blocks.

A PBD with no blocks equal to \(X\) is a proper PBD.

A PBD with all blocks equal to \(X\) is a trivial PBD.
A Simple Family of BIBD's

$k = 2, \lambda = 1$

If a BIBD has the parameters $k = 2$ and $\lambda = 1$, then it is easily calculated that $r = v - 1$ and $b = v(v - 1)/2$. This means that the blocks of the design are just all possible pairs of varieties, i.e., the set of blocks is the set of all 2-subsets of $X$.

If we interpret the varieties of the design as being vertices and the blocks as being edges, then a design with these parameters is a complete graph on $v$ vertices. From the viewpoint of design theory, these are not very interesting designs even though they are an important class of graphs.
Fisher's Inequality

**Theorem** - [Fisher's Inequality] \( \text{In a } (v,k,\lambda)-\text{design, } b \geq v. \)

**Proof**: Suppose that \( b < v \) and let \( A \) be the incidence matrix of the design. We can add \( v - b \) columns of 0's to \( A \) to get a \( v \times v \) matrix \( B \). Since these extra columns of 0's will not alter the inner products, we must have \( AA^T = BB^T \). By taking determinants we see that,

\[
\det(AA^T) = \det(BB^T) = (\det B)(\det B^T) = 0
\]

since \( \det B = 0 \) due to the columns of 0's. Now by the matrix identity we have:

\[
det(AA^T) = \det \begin{pmatrix}
    r & \lambda & \lambda & \cdots & \lambda \\
    \lambda & r & \lambda & \cdots & \lambda \\
    \lambda & \lambda & r & \cdots & \lambda \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \lambda & \lambda & \lambda & \cdots & r
\end{pmatrix}
\]
Fisher's Inequality

**Theorem** - [Fisher's Inequality]  In a \((v,k,\lambda)\)-design, \(b \geq v\).

**Pf (cont)**: We can evaluate this determinant by subtracting the first column from each of the other columns and then adding each row to the first row to obtain the following:

\[
\det \left( AA^T \right) = \det \begin{pmatrix}
    r + \lambda(v-1) & 0 & 0 & \cdots & 0 \\
    \lambda & r - \lambda & 0 & \cdots & 0 \\
    \lambda & 0 & r - \lambda & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    \lambda & 0 & 0 & \cdots & r - \lambda
\end{pmatrix}.
\]

So we have that,

\[
\det \left( AA^T \right) = [r + \lambda(v-1)](r-\lambda)^{v-1}.
\]

But since \(r,v\) and \(\lambda\) are all positive, \(r + \lambda(v-1) > 0\) and since \(k < v\) we have that \(r > \lambda\), so this product on the right can not equal 0 contradiction.
Fisher's Inequality

This result is very important, so we shall give another proof here and you are asked to provide others in the homework.

Alternate Proof of Fisher's Inequality:
Let A be the incidence matrix of the BIBD. Let $s_j$ be the $j^{\text{th}}$ row of $A^T$. Note that the $s_j$'s are all vectors in $\mathbb{R}^v$ and there are $b$ of them. Let $S$ be the span $\langle s_j \mid 1 \leq j \leq b \rangle$, that is, all the linear combinations (over $\mathbb{R}$) of the $s_j$'s. If we can show that $S = \mathbb{R}^v$, then since a spanning set can not be smaller than a basis, we must have $b \geq v$, giving the result.

We will prove that $S$ is a spanning set by showing that all the standard basis elements $e_i$ are in $S$. 
Fisher's Inequality

Alternate Proof of Fisher's Inequality:

Observe that

$$\sum_{j=1}^{b} s_j = (r, \cdots, r),$$

so

$$\sum_{j=1}^{b} \frac{1}{r} s_j = (1, \cdots, 1).$$

For a fixed $i$, summing only those rows $s_j$ which have a 1 in the $i^{th}$ column of $A^T$ gives

$$\sum_{\text{special}} s_j = (r - \lambda) e_i + (\lambda, \cdots, \lambda) = (r - \lambda) e_i + \sum_{j=1}^{b} \frac{\lambda}{r} s_j.$$
Fisher's Inequality

Alternate Proof of Fisher's Inequality (cont):

Since $\lambda(v-1) = r(k-1)$ and $v > k$, it follows that $r > \lambda$, and so $r - \lambda > 0$, and we may solve this equation for $e_i$ to get

$$e_i = \sum_{special} \frac{1}{r-\lambda} s_j - \sum_{j=1}^{b} \frac{\lambda}{r(r-\lambda)} s_j.$$

Thus, $e_i$ for all $i$, is a linear combination of the $s_j$'s and so is in $S$.

Fisher's inequality holds more generally for all nontrivial (not all blocks are X) PBD's, but these proofs don't quite work in that generality.
New BIBD's from old ones

There are two very easy ways to get new BIBD's from given ones.

**Theorem (Sum Construction):** Suppose there exists a \((v,k,\lambda_1)\)-BIBD and a \((v,k,\lambda_2)\)-BIBD. Then there exists a \((v,k,\lambda_1 + \lambda_2)\)-BIBD.

**Corollary.** Suppose there exists a \((v,k,\lambda)\)-BIBD. Then there exists a \((v,k,s\lambda)\)-BIBD for all positive integers \(s\).

**Pf:** Build the two BIBD's on the same set \(X\) then union all the blocks. For the corollary, do this with \(s\) copies of the original BIBD.
New BIBD's from old ones

Theorem (Block Complementation) If there exists a \((v,b,r,k,\lambda)\)-BIBD with \(k \leq v-2\), then there exists a \((v,b,b-r,v-k,b-2r+\lambda)\)-BIBD.

\textbf{Pf:} Take as new blocks the complements \((X-B)\) for each block \(B\) of the old design. Use PIE to get the new \(\lambda\) value.

And a method that doesn't always work.

If a \((v,k,\lambda)\)-BIBD has incidence matrix \(A\), then the design obtained from incidence matrix \(A^T\) called the \textbf{dual design}, won't even be a PBD unless \(v = b\) (Fisher's inequality).