Quadratic Sets

4.1 Fundamental Definitions
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Tangents

**Def:** Let \( Q \) be a set of points of the projective space \( \mathbf{P} \). A line \( g \) is a *tangent* of \( Q \) if either \( g \) has just one point in common with \( Q \) or each point of \( g \) is contained in \( Q \).

For each point \( P \) of \( Q \) let the set \( Q_p \) consist of the point \( P \) and all the points \( X \neq P \) such that the line \( XP \) is a tangent of \( Q \). One calls \( Q_p \) the *tangent space* of \( Q \) at \( P \).
Quadratic Sets

We call the set $Q$ a *quadratic set* of $\mathbf{P}$ if it satisfies:

1. **If-three-then-all axiom.** Any line $g$ that contains at least three points of $Q$ is totally contained in $Q$. In other words, any line not contained in $Q$ can meet $Q$ in at most 2 points.

2. **Tangent-space axiom.** For any point $P$ of $Q$, its tangent space $Q_P$ is either a hyperplane or all of $\mathbf{P}$. 
Examples of Quadratic Sets

1. The empty set and any subspace of $P$ is a quadratic set.
2. The union of two planes is a quadratic set.
3. In 3-space, hyperbolic quadrics and cones over a quadratic set are quadratic sets.
Examples of Quadratic Sets

4. In Euclidean geometry, spheres in 3 dimensions and conics in the plane are quadratic sets.
Lemma 4.1.1: Let $Q$ be a quadratic set of $P$, and let $U$ be a subspace of $P$. Then the set $Q' := Q \cap U$ of the points of $Q$ in $U$ is a quadratic set of $U$. Moreover, we have

$$Q'_P = Q_P \cap U$$

for all points $P \in Q'$. 

*Pf:* Clearly $Q'$ satisfies the if 3 then all axiom. Let $P \in Q'$. We have

$$Q'_P = \{P\} \cup \{X | XP \text{ is a tangent, } X \in U\} = Q_P \cap U.$$ 

Since $Q_P$ is a subspace of $P$ of dimension at least $d-1$, $Q_P \cap U$ is a subspace of $U$ of dimension at least $\dim(U) - 1$. 

\[\square\]
The Radical

**Def:** Let $Q$ be a quadratic set of $\mathbb{P}$. The *radical* of $Q$ is the set $\text{rad}(Q)$ of all points $P$ in $Q$ with the property that $Q_P = P$.

We say that $Q$ is *nondegenerate* if $\text{rad}(Q) = \emptyset$, that is, if for each point $P$ of $Q$, $Q_P$ is a hyperplane.

*Example:*
A sphere in Euclidean space is a nondegenerate quadratic set, while a cone is degenerate since its radical consists of the vertex of the cone.
Theorem 4.1.2: Let $Q$ be a quadratic set of $\mathbb{P}$.

a. The radical of $Q$ is a (linear) subspace of $\mathbb{P}$.

b. Let $U$ be a complement of rad($Q$) (that is, a subspace $U$ such that $U \cap \text{rad}(Q) = \emptyset$ and $<U, \text{rad}(Q)> = \mathbb{P}$). Then $Q' := Q \cap U$ is a nondegenerate quadratic set of $U$.

c. $Q$ can be described as follows: $Q$ consists of all points that lie on lines that join a point of rad($Q$) with a point of $Q' = Q \cap U$. 
Theorem 4.1.2: Let $Q$ be a quadratic set of $\mathbb{P}$. a) The radical of $Q$ is a (linear) subspace of $\mathbb{P}$.

*Proof* (a) Let $P, P'$ in $\text{rad}(Q)$. Let $P''$ be a third point on $PP'$. We have to show that $P''$ also lies in $\text{rad}(Q)$, that is, $Q_{p''}$ contains all the points of $\mathbb{P}$.

Since every line through a point of the radical is a tangent line, $PP'$ is a tangent line containing at least two points of $Q$. Thus, all the points of $PP'$ are in $Q$. Now assume that $Q_{p''}$ is only a hyperplane (which must contain the line $PP'$). Consider a line $g$ through $P''$ which is not contained in $Q_{p''}$. The line $g$ is not a tangent line, so it must contain precisely one other point $R$ in $Q$. The lines $PR$ and $P'R$ are tangent lines with more than one point of $Q$, so they are entirely contained in $Q$. Let $T$ be a point of $PR$ different from $P$ and $R$.

$P'T$ is a tangent with 2 points in $Q$, so is contained in $Q$. Thus, the point $T' = g \cap P'T$ is in $Q$, and since $T'$ can not be $R$ or $P''$, we obtain a contradiction.
Theorem 4.1.2: Let $Q$ be a quadratic set of $P$.

a. The radical of $Q$ is a (linear) subspace of $P$.

b. Let $U$ be a complement of $\text{rad}(Q)$ (that is, a subspace $U$ such that $U \cap \text{rad}(Q) = \emptyset$ and $<U, \text{rad}(Q)> = P$). Then $Q' := Q \cap U$ is a nondegenerate quadratic set of $U$.

c. $Q$ can be described as follows: $Q$ consists of all points that lie on lines that join a point of $\text{rad}(Q)$ with a point of $Q' = Q \cap U$.

\textbf{Pf (cont.):} (b) Assume that $Q'$ is degenerate. Then there is a point $P$ in $Q'$ so that all lines of $U$ through $P$ are tangent lines. Any line through $P$ and a point of $\text{rad}(Q)$ is also a tangent line. Since $<U, \text{rad}(Q)> = P$, every line through $P$ not contained in $U$ contains a point of $\text{rad}(Q)$, so all lines through $P$ are tangent lines and $P$ in $\text{rad}(Q)$, contradicting the choice of $U$.

(c) Any point of $Q$ not in $U$ or $\text{rad}(Q)$ is on a line joining a point of $U$ with a point of $\text{rad}(Q)$. This line is a tangent contained in $Q$, so the point of $U$ on it is in $Q'$.
Lemma 4.1.3

Remark: In view of this theorem, we can restrict ourselves to the study of nondegenerate quadratic sets.

Lemma 4.1.3: Let $Q$ be a nondegenerate quadratic set of $P$. For any two distinct points $P$ and $R$ in $Q$, we have $Q_p \neq Q_R$. In other words, the quadratic set that is induced by $Q$ in a tangent space $Q_p$ has a radical that consists of just one point, namely $P$.

Pf: Suppose that for distinct points $P$ and $R$ we have $Q_p = Q_R = H$ a hyperplane. If $Q'$ is the quadratic set induced by $Q$ in $H$ ($= Q \cap H$), then $\text{rad}(Q')$ contains at least the two points $P$ and $R$. Since the radical is a subspace, all the points of the line $PR$ are contained in $\text{rad}(Q')$. 
Lemma 4.1.3

Lemma 4.1.3: Let $Q$ be a nondegenerate quadratic set of $\mathbb{P}$. For any two distinct points $P$ and $R$ in $Q$, we have $Q_P \neq Q_R$. In other words, the quadratic set that is induced by $Q$ in a tangent space $Q_p$ has a radical that consists of just one point, namely $P$.

Pf (cont): Since $Q$ is nondegenerate, each line through $P$ which is not in $H$ is not a tangent, so contains precisely one other point of $Q$. In particular, there is a point $S$ of $Q$ not in $H$. Consider the tangent space $Q_S$ at $S$; since it is a hyperplane it intersects $PR$ in some point $T$.

$T$ is in $\text{rad}(Q')$ (since $PR$ is in $\text{rad}(Q')$), so $Q_T$ contains $H$. The line $ST$ lies in $Q_S$, so, in particular each point of $ST$ lies in $Q$. Thus, $ST$ is a tangent line through $T$, and therefore in $Q_T$. So, $Q_T$ contains $H$ and the point $S$ not in $H$, which implies that $Q_T = P$, and so $Q$ is degenerate, a contradiction. $\square$
Nondegenerate Quadratics

Lemma 4.1.4: Let $Q$ be a nondegenerate quadratic set of $\mathbf{P}$.

a. If $P$ is a point of $Q$ and $W$ is a complement of $P$ in $Q_p$, then $Q' := Q \cap W$ is a nondegenerate quadratic set of $W$.

b. If $H$ is a hyperplane that is not a tangent hyperplane (tangent space) then $Q' := Q \cap H$ is a nondegenerate quadratic set of $H$.

\textit{Pf:} (a) By Lemma 4.1.3, $\text{rad}(Q \cap Q_p) = \{P\}$. Thus, by Theorem 4.1.2 (b), $Q'$ is nondegenerate.

(b) Assume that there is a point $X$ in $\text{rad}(Q')$. Then $Q'_X = H$. Since $H$ is not a tangent hyperplane, we have $Q'_X \neq H$. This contradicts Lemma 4.1.1, which says that $Q'_X = Q_X \cap H$. \(\square\)
The Index of a Quadratic Set

**Def:** Given a quadratic set \( Q \), any subspace contained in \( Q \) is called a \( Q \)-subspace.

Let \( t-1 \) be the maximum dimension of a \( Q \)-subspace of a quadratic set \( Q \). Then the integer \( t \) is called the **index** of \( Q \). The \( Q \)-subspaces of dimension \( t-1 \) are also called **maximal** \( Q \)-subspaces.

The index is the algebraic (vector space) dimension of a maximal \( Q \)-subspace.

**Examples:** A cone and a hyperboloid in 3-dimensional Euclidean space have index 2 since they contain lines but no planes. The quadratic set consisting of the union of two planes in a projective space has index 3. Any quadratic set which does not contain a line has index 1.
The Index of a Quadratic Set

Lemma 4.2.1: Let $Q$ be a nondegenerate quadratic set of index $t$ in $P$. Then each point of $Q$ is on a maximal $Q$-subspace. More precisely: if $P$ is a point of $Q$ outside a $(t-1)$-dimensional $Q$-subspace $U$, then there is a $(t-1)$-dimensional $Q$-subspace $U'$ through $P$ which intersects $U$ in a $(t-2)$-dimensional subspace.

Remark: An important case of the above lemma is when $t = 2$: Through each point of $Q$ not on a $Q$-line $g$, there passes a $Q$-line which intersects $g$.

$Pf$: The tangent hyperplane $Q_p$ at $P$ intersects $U$ in a subspace $V$ of dimension $t-2$. It follows that each line $PX$ with $X \in V$ is a tangent and therefore contained in $Q$. Thus, $U' = \langle P, V \rangle$ is a $(t-1)$-dimensional $Q$-subspace. \qed
Lemma 4.2.2: Let $Q$ be a quadratic set in $P$. Let $S$ be a subset of $Q$ with the property that the line through any two points of $S$ is a $Q$-line. Then $\langle S \rangle$ is a $Q$-subspace.

\textit{Pf:} We can assume wlog that $S$ is finite. Since any spanning set contains a finite basis, there is a finite subset $S_0$ such that $\langle S_0 \rangle = \langle S \rangle$. It is therefore sufficient to show that $\langle S_0 \rangle$ is a $Q$-subspace.

We now use induction on the size of $S$. For $|S| = 0, 1$ or $2$ the assertion is trivial. So suppose $|S| > 2$ and assume that the assertion is true for each set with $|S|-1$ elements. Consider a point $P \in S$. By induction $V = \langle S \setminus \{P\} \rangle$ is a $Q$-subspace. We may assume $P \not\in V$. By hypothesis, for each point $R$ in $S \setminus \{P\}$, the line $RP$ is a $Q$-line. Since these lines generate the subspace $\langle S \setminus \{P\}, P \rangle = \langle S \rangle$, the tangent space of $Q$ at the point $P$ contains the subspace $\langle S \rangle = \langle V, P \rangle$. Therefore all lines $XP$ with $X$ in $V$ are contained in $Q$. It follows that $\langle S \rangle = \langle V, P \rangle \subseteq Q$. □
The Index of a Quadratic Set

**Theorem 4.2.3:** Let $Q$ be a quadratic set in a $d$-dimensional projective space $P$, and let $U$ be a maximal $Q$-subspace. If $Q$ is nondegenerate, then there is a maximal $Q$-subspace that is skew to (disjoint from) $U$.

*Proof (Pf)*: Let $t$ be the index of $Q$. We will prove the following more general result which implies this theorem: *If $j \in \{-1, 0, ..., t-2\}$ then there is a maximal $Q$-subspace $U_j$ such that $\dim(U \cap U_j) = j$.*

We proceed by induction on $j$. If $j = t-2$ then the assertion follows from Lemma 4.2.1. So now suppose that $0 \leq j \leq t-2$, and let $U'$ be a maximal $Q$-subspace with $\dim(U \cap U') = j$. We will construct a maximal $Q$-subspace $U''$ with $\dim(U \cap U'') = j-1$.

Observe that there exists a point $P$ in $Q$ such that $\langle U \cap U', P \rangle$ is not a $Q$-subspace. Otherwise, any point of $U \cap U'$ would be in $\text{rad}(Q)$ and since $\dim(U \cap U') \geq 0$ this contradicts $Q$ being nondegenerate.
The Index of a Quadratic Set

**Theorem 4.2.3:** Let $Q$ be a quadratic set in a $d$-dimensional projective space $P$, and let $U$ be a maximal $Q$-subspace. If $Q$ is nondegenerate, then there is a maximal $Q$-subspace that is skew to (disjoint from) $U$.

**Pf(cont.):** By Lemma 4.2.1 there is a maximal $Q$-subspace $W$ of $P$ through $P$ intersecting $U'$ in a subspace of dimension $t-2$. We claim that $W$ satisfies the claim. Since $U \cap U'$ is not contained in $W$ we have that $\dim(W \cap U \cap U') = j-1$. We will now show that $W \cap U = W \cap U \cap U'$.

Assume that there is a point $X$ in $W \cap U$ with $X \notin U'$. Then the set $S = (W \cap U') \cup (U \cap U') \cup \{X\}$ satisfies the hypothesis of Lemma 4.2.2. Hence $M = \langle S \rangle$ is a $Q$-subspace. This subspace contains the hyperplane $W \cap U'$ of $W$ and the point $X$ in $W \setminus U'$, and hence the whole subspace $W$. Thus $M = W$. So we would have $U \cap U' \subseteq M = W$, contradicting the choice of $P$. $\square$
The Index of a Quadratic Set

**Theorem 4.2.4:** Let $Q$ be a nondegenerate quadratic set of index $t$ in a $d$-dimensional projective space $P$. If $d$ is even then $t \leq d/2$; if $d$ is odd then $t \leq (d+1)/2$.

*Pf:* By the preceding theorem there are two skew $(t-1)$-dimensional $Q$-subspaces $U$ and $U'$. They satisfy

\[ \dim(P) \geq \dim(U) + \dim(U') - \dim(U \cap U'), \]

so $d \geq 2(t-1) + 1$. \qed