The Clebsch Graph

The Clebsch graph is a strongly regular Quintic graph on 16 vertices and 40 edges with parameters (γ, k, λ, μ) = (16, 5, 0, 2). It is also known as the Greenwood-Gleason Graph.

The Clebsch Graph is a member of the following Graph families

1. **Strongly regular**

2. **Quintic graph :** A graph which is 5-regular.

3. **Non Planar and Hamiltonian :**

   A Hamiltonian graph, is a graph possessing a Hamiltonian circuit which is a graph cycle (i.e., closed loop) in a graph that visits each node exactly once

4. **Vertex Transitive Graph :**

   A vertex – transitive graph is a Graph G such that, given any two vertices v_1 and v_2 of G, there is some automorphism

   $$f: V(G) \to V(G)$$
   $$\text{such that}$$
   $$f(v_1) = f(v_2)$$

   It is a way of mapping the object to itself while preserving all of its structure

5. **Integral graph :** A graph whose spectrum consists entirely of integers is known as an integral graph.

Parallel Definition of the Clebsch Graph

The Clebsch graph is an integral graph and has graph spectrum $(-3)^5(1)^{10}(5)^1$

Integral Graph and explanation of the terms used in its definitions

A. The adjacency matrix :

The adjacency matrix A of a graph G is the matrix with rows and columns indexed by the vertices, with Axy = 1 if x and y are adjacent, and Axy =0 otherwise. The adjacency spectrum of G is the spectrum of A, that is, its multiset of eigenvalues. A number λ is called an eigenvalue of a matrix M if there is a nonzero vector u such that Mu = λu. The vector u is called an eigenvector. As the adjacency matrix is symmetric, the (adjacency) spectrum of a graph consists of real numbers. An undirected graph has a symmetric adjacency matrix and therefore has real eigenvalues and a complete set of orthonormal eigenvectors.

B. Eigen Values :

The eigenvalues of a graph are defined as the eigenvalues of its adjacency matrix. The set of eigenvalues of a graph is called a graph spectrum.

C. The Spectrum of the Graph :

The set of all eigenvalues of the adjacency matrix is called the spectrum of the graph.

D. **Integral Graph :**

A graph whose spectrum consists entirely of integers is known as an integral graph.

Characteristic Properties of a Clebsch Graph:

1. The chromatic number of the Clebsch graph is 4.
2. It is the local graph of the Schlafli graph. The sub graph of the Schlafli graph on the set of non-neighbors of a vertex is the Clebsch graph.
3. If a loop is added to each vertex of the Clebsch Graph, the resulting adjacency matrix is equivalent to a 2-(16, 6, 2) block design.
4. The complement of the Clebsch graph is a strongly regular graph. It has least Eigenvalue -2. It can be represented in E_8 .
5. In fact , any Strongly Regular Graph with least eigenvalue -2 is one of the following :

   $T(m), m \geq 5$;

   $L_2(m), m \geq 3$;

   $CP(m), m \geq 2$;

   The Peterson Graph;

   The complement of the Clebsch Graph ;

   The complement of the Schlafli graph;

   The Shrikhande Graph;

   The three Chang Graphs;

6. The subgraph on the non-neighbors of a point in the Clebsch Graph is the Petersen graph.

7. The order of the largest independent set in a Clebsch graph is 5

8. Clebsch Graph is a triangle free Graph.

9. Every triangle-free planar graph has a homomorphism to Clebsch graph

Construction of Clebsch Graph:

1. Let the total number of vertices be $V(X) = z \cup V_1 \cup V_2$

2. The vertices in V_1 are (1, 2, 3, 4, 5). They form an independent set.

3. The vertices in V_2 are denoted by 2-sized subsets of the set (1, 2, 3, 4, 5). The two subsets are adjacent if they are disjoint (i.e. if they have no co-ordinate in common).

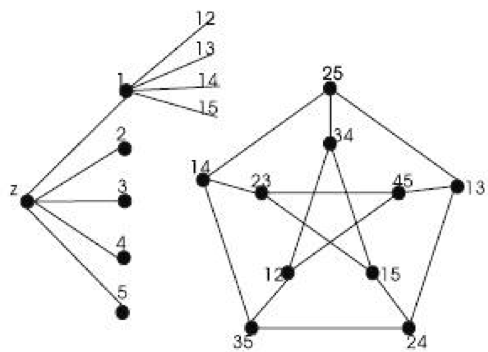
4. The vertices in V_2 induce a copy of the Petersen graph.

5. The vertex z is adjacent to all vertices in V1.

6. The vertices in V1 labeled i:(i=1,2,3,4,5) is adjacent to those vertices in V2 whose label contains i as one of the co-ordinates.

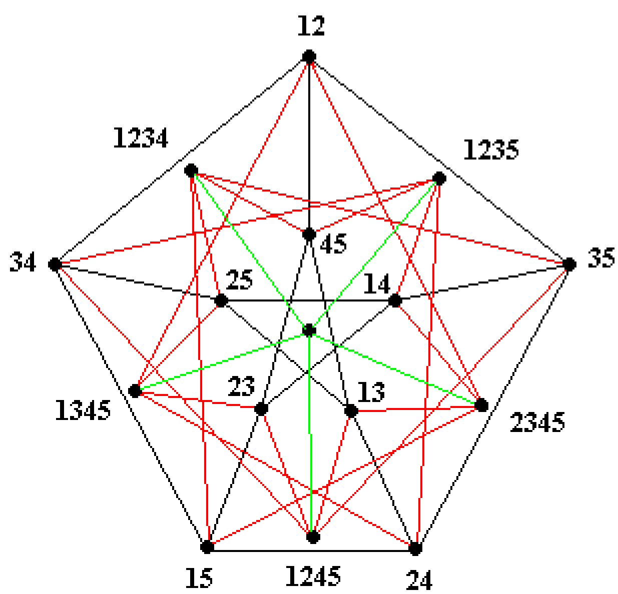
Vertex 1 is adjacent to 12,13,14,15
Vertex 2 is adjacent to 12,23,24,25
Vertex 3 is adjacent to 13,23,34,35
Vertex 4 is adjacent to 14,24,34,45

Vertex 5 is adjacent to 15,25,35,45



Another construction of the Clebsch Graph :

The Clebsch graph has as vertices all subsets of 1, 2,3,4,5 of even cardinality; two vertices are adjacent if their symmetric difference has cardinality 4.



Clebsch Graph

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