Dealing with the Infinite

Part II

How many points in a square?

Cantor pressed forward, exchanging letters throughout with Dedekind. The next question he asked himself, in January 1874, was whether the unit square could be mapped into a line of unit length with a 1-1 correspondence of points on each. In a letter to Dedekind dated 5 January 1874 he wrote :-

Can a surface (say a square that includes the boundary) be uniquely referred to a line (say a straight line segment that includes the end points) so that for every point on the surface there is a corresponding point of the line and, conversely, for every point of the line there is a corresponding point of the surface? I think that answering this question would be no easy job, despite the fact that the answer seems so clearly to be "no" that proof appears almost unnecessary.

1874

Cantor with his wife, Vally, about 1880.
In the possession of Egbert Schneider.

The year 1874 was an important one in Cantor's personal life. He became engaged to Vally Guttmann, a friend of his sister, in the spring of that year. They married on 9 August 1874 and spent their honeymoon in Interlaken in Switzerland where Cantor spent much time in mathematical discussions with Dedekind.

1874 - 1877

Cantor continued to correspond with Dedekind, sharing his ideas and seeking Dedekind's opinions, and he wrote to Dedekind in 1877 proving that there was a 1-1 correspondence of points on the interval [0, 1] and points in p-dimensional space. Cantor was surprised at his own discovery and wrote:-

I see it, but I don't believe it!

1877

Of course this had implications for geometry and the notion of dimension of a space. A major paper on dimension which Cantor submitted to Crelle's *Journal* in 1877 was treated with suspicion by Kronecker, and only published after Dedekind intervened on Cantor's behalf. Cantor greatly resented Kronecker's opposition to his work and never submitted any further papers to Crelle's *Journal*.

The paper on dimension which appeared in Crelle's *Journal* in 1878 makes the concepts of 1-1 correspondence precise. The paper discusses denumerable sets. It studies sets of equal power, i.e. those sets which are in 1-1 correspondence with each other. Cantor also discussed the concept of dimension and stressed the fact that his correspondence between the interval [0,1] and the unit square was not a continuous map.

Dimension

Cantor's original "proof" of the equivalence of the set of points of a unit square and the points of the unit interval went like this:

To each point of the unit square $(x,y) = (0.a_1a_2a_3\ldots, 0.b_1b_2b_3\ldots)$ associate the number $z = 0.a_1b_1a_2b_2a_3b_3\ldots$. This is easily seen to be a one-to-one function. However, as Dedekind pointed out to Cantor, it is not onto the interval [0,1]. The problem arises because of the non-uniqueness of the decimal representations. Cantor had chosen to take, when there were two representations for the same number, the form without repeating 0's (except for the number 0 itself). With this convention, a number like $z = 0.12304050607\ldots$ could never be obtained since it would come from $x = 0.134567\ldots$ and $y = 0.200000\ldots$ and this last is a forbidden form. Cantor fixed the proof, but was unhappy about losing the simple formulation.

Dimension

An easy fix (not due to Cantor) which keeps the simple spirit of the original, is to view the decimal expansions not as a sequence of digits, but rather as a sequence of "molecules" where a molecule consists of a nonzero digit and all the preceding 0's back to the previous nonzero digit. Thus, a number like

$$w = 0.12003000450600000789....$$

would be broken up into molecules $a_1 = 1$, $a_2 = 2$, $a_3 = 003$, $a_4 = 0004$, $a_5 = 5$, $a_6 = 06$, $a_7 = 000007$, $a_8 = 8$ and $a_9 = 9$... .

With this idea, the troublesome number $z = 0.12304050607...$ would come from $x = 0.130507...$ and $y = 0.20406...$. The function is now a bijection.

Dimension

Cantor clearly pointed out that the function used in this result was not a continuous one. He believed that there was a connection between continuity and dimension which was of a higher order than mere equivalence. Soon after the paper was published several mathematicians established that for dimensions no greater than 3, a continuous function between spaces of different dimensions could not be one-to-one. However, their arguments ran into difficulties for higher dimensions. Cantor reentered the fray with a paper in 1889 which contained a fallacious proof of the result for higher dimensions. The errors in this proof were not found for 20 years. The problems were subtle and it was generally accepted that Cantor had given a correct proof of the result, so the issue was considered settled for this interval.

1879 - 1884

Between 1879 and 1884 Cantor published a series of six papers in *Mathematische Annalen* designed to provide a basic introduction to set theory. Klein may have had a major influence in having *Mathematische Annalen* published them. However there were a number of problems which occurred during these years which proved difficult for Cantor. Although he had been promoted to a full professor in 1879 on Heine's recommendation, Cantor had been hoping for a chair at a more prestigious university. His long standing correspondence with Schwarz ended in 1880 as opposition to Cantor's ideas continued to grow and Schwarz no longer supported the direction that Cantor's work was going. Then in October 1881 Heine died and a replacement was needed to fill the chair at Halle.

1879 - 1884

Cantor drew up a list of three mathematicians to fill Heine's chair and the list was approved. It placed Dedekind in first place, followed by Heinrich Weber and finally Mertens. It was certainly a severe blow to Cantor when Dedekind declined the offer in the early 1882, and the blow was only made worse by Heinrich Weber and then Mertens declining too. After a new list had been drawn up, Wangerin was appointed but he never formed a close relationship with Cantor. The rich mathematical correspondence between Cantor and Dedekind ended later in 1882.

Grundlagen

Almost the same time as the Cantor-Dedekind correspondence ended, Cantor began another important correspondence with Mittag-Leffler. Soon Cantor was publishing in Mittag-Leffler's journal *Acta Mathematica* but his important series of six papers in *Mathematische Annalen* also continued to appear. The fifth paper in this series *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* was also published as a separate monograph and was especially important for a number of reasons. Firstly Cantor realized that his theory of sets was not finding the acceptance that he had hoped and the *Grundlagen* was designed to reply to the criticisms.
Secondly :-

The major achievement of the Grundlagen was its presentation of the transfinite numbers as an autonomous and systematic extension of the natural numbers.

Grundlagen

Cantor himself states quite clearly in the paper that he realises the strength of the opposition to his ideas:-

... I realise that in this undertaking I place myself in a certain opposition to views widely held concerning the mathematical infinite and to opinions frequently defended on the nature of numbers.

Transfinite Numbers

In order to gain acceptance for his new ideas, Cantor had to ground them on concepts that were already familiar. He did this by using the concept of order.

A relation on a set is called a ***partial order*** if it is *reflexive*, *antisymmetric* and *transitive*. That is, if we use the symbol "≤" to denote the relation on the set X,

1) $a \le a$ for all $a \in X$,
2) if $a \le b$ and $b \le a$ then $a = b$ for all $a,b \in X$, and
3) if $a \le b$ and $b \le c$ then $a \le c$ for all $a,b,c \in X$.

Two partially ordered sets (***posets***) $(X, \le_1)$ and $(Y, \le_2)$ are ***similar*** if there exists a bijection $f: X \to Y$ which preserves the order,

$$x_1 \le_1 x_2 \text{ iff } f(x_1) \le_2 f(x_2).$$

Such a function is called a ***similarity***.

Transfinite Numbers

Similarity of posets is an equivalence relation on any set of posets, and the equivalence classes are called ***order types***.

In order to show that two posets are not of the same order type, we often make use of the obvious fact that two similar sets either both have a first (last) element or neither does. Thus, the equivalent sets $\mathbb{N} = <1,2,3, ...>$ and $\mathbb{N}^* = <...,3,2,1>$ are not similar and so do not have the same order type. The order type of $\mathbb{N} = <1,2,3, ...>$ is denoted by ω.

Finite sets of the same size all have the same order type, so for them the concept of order type does not refine the idea of equivalence. Thus, we use the size of the finite set to denote both its cardinality and its order type. So, $A = <3,5,2,1>$ has order type 4.

Transfinite Numbers

A poset is ***well-ordered*** iff every non-empty subset has a first element. The natural numbers with their usual ordering is a well-ordered set. The reals with their usual ordering are not well-ordered. $\mathbb{N}^* = \langle \ldots,3,2,1 \rangle$ is not well-ordered.

If a poset is well-ordered, then all the posets with the same order type are also well-ordered. Thus, the following definition is not ambiguous. We define the order type of a well-ordered set to be an ***ordinal number***.

As every finite set (with any ordering) is well-ordered, every natural number n is an ordinal number. We also have that ω is an ordinal number which is not a natural number.

Transfinite Numbers

Ordinal numbers can be added and multiplied (also subtracted and divided – but these operations are more difficult to define and we won't go there).

If α and β are ordinal numbers, the sum $\alpha + \beta$ is the order type of the well-ordered set <A;B> where A is some set of order type α and B some disjoint set of order type β, and the notation <A;B> means that every element of the set B is greater than every element of the set A.

Thus, $1 + \omega$ is the order type of the poset <0; 1,2,3,...>, and this is clearly ω (i.e., $1 + \omega = \omega$). On the other hand, $\omega + 1$ would be the order type of the poset <1,2,3, ... ; 0> and this is not ω, since it has a last element. This example shows that addition of ordinal numbers is not generally commutative. As another example, $\omega + \omega$ would be the order type of <1,3,5, ... ; 2,4,6, ...>

Transfinite Numbers

If α and β are ordinal numbers, the product $\alpha\beta$ is the order type of the well-ordered set $B \times A$ where A is some set of order type α and B some set of order type β, and the ordered pairs of $B \times A$ are ordered lexicographically (dictionary order).

Thus, 2ω is the order type of the poset we construct from the sets $A = \langle a,b \rangle$ and $B = \langle 1,2,3, \ldots \rangle$, that is $\langle (1,a),(1,b),(2,a),(2,b),(3,a),(3,b),\ldots \rangle$ and this is ω (i.e., $2\omega = \omega$). On the other hand, $\omega 2$ is the order type of the poset $\langle (a,1),(a,2),(a,3), \ldots ; (b,1),(b,2),(b,3) \rangle$ and this is $\omega + \omega$. This example shows that multiplication of ordinal numbers is also not generally commutative.

Since exponentiation is repeated multiplication, we have for example $\omega^2 = \omega\omega = \omega + \omega + \omega + \ldots$. A set of this order type is:

$$\langle 1,2,3,\ldots ; 1/2, 3/2, 5/2, \ldots ; 1/3, 2/3, 4/3, \ldots ; \ldots \rangle.$$

Transfinite Numbers

Although ordinal numbers do not satisfy all the familiar arithmetic properties, both addition and multiplication are associative and the left (but not the right) distributive law of multiplication over addition holds.

It is natural to introduce a partial order on ordinal numbers, by defining $\alpha \leq \beta$ iff A is similar to a subset of B where α is the ordinal number of A and β the ordinal number of B.

With this definition of partial order, it can be shown that:

Theorem: *Every set of ordinal numbers is a well-ordered set.*

Consequently, any set of ordinal numbers **has** an ordinal number. Moreover, it can be shown that every ordinal number **is** the ordinal number of the set of all ordinal numbers which precede it.

For finite ordinals, this just says that the ordinal number of $<0,1,2,\ldots, n-1>$ is n. But it also says that $\omega + 1$ is the ordinal number of the set $<0,1,2,\ldots , \omega>$.

Transfinite Numbers

It should now be clear that ω is the smallest ordinal number which is not finite. It's definition as the order type of the well ordered natural numbers, does not require any preconceived idea of infinity, and it forces one to view the natural numbers as a “complete” set.

These transfinite numbers that Cantor had firmly established the existence of, could now be used to further investigate the concept of power (cardinality) of a set that he had earlier introduced.

We should mention a result which was not yet available to Cantor.

Well-Ordering Theorem (Zermelo, 1906) : Every set can be well-ordered.

This result is logically equivalent to the Axiom of Choice, but it does say that any set has at least one associated ordinal number.

Depression

At the end of May 1884 Cantor had the first recorded attack of depression. He recovered after a few weeks but now seemed less confident. He wrote to Mittag-Leffler at the end of June :-

... I don't know when I shall return to the continuation of my scientific work. At the moment I can do absolutely nothing with it, and limit myself to the most necessary duty of my lectures; how much happier I would be to be scientifically active, if only I had the necessary mental freshness.

At one time it was thought that his depression was caused by mathematical worries and as a result of difficulties of his relationship with Kronecker in particular. Recently, however, a better understanding of mental illness has meant that we can now be certain that Cantor's mathematical worries and his difficult relationships were greatly magnified by his depression but were not its cause.

Depression

...Cantor was often misunderstood, not only by his contemporaries, but by later biographers and historians as well. ... In his own day Cantor was regarded as an eccentric, if exciting, man, who apparently stimulated interest wherever he went, particularly among younger mathematicians. But it was the mathematician and historian E.T. Bell who popularized the portrait of a man whose problems and insecurities stemmed from Freudian antagonisms with his father and whose relationship with his archrival Leopold Kronecker was exacerbated because both men were Jewish [“There is no more vicious academic hatred than that of one Jew for another when they disagree on purely scientific matters. When two intellectual Jews fall out they disagree all over, throw reserve to the dogs, and do everything in their power to cut one anothers' throats or stab one another in the back” - Bell (1939)] ... Equally unreliable are Bell's assertions about Cantor's mental illness.

- Dauben, *Georg Cantor*, 1979

Reconciliation

After this mental illness of 1884 :-
... he took a holiday in his favourite Harz mountains and for some reason decided to try to reconcile himself with Kronecker. Kronecker accepted the gesture, but it must have been difficult for both of them to forget their enmities and the philosophical disagreements between them remained unaffected.
Mathematical worries began to trouble Cantor at this time, in particular he began to worry that he could not prove the continuum hypothesis, namely that the order of infinity of the real numbers was the next after that of the natural numbers. In fact he thought he had proved it false, then the next day found his mistake. Again he thought he had proved it true only again to quickly find his error.

1885

All was not going well in other ways too, for in 1885 Mittag-Leffler persuaded Cantor to withdraw one of his papers from *Acta Mathematica* when it had reached the proof stage because he thought it "*... about one hundred years too soon*". Cantor joked about it but was clearly hurt:-

Had Mittag-Leffler had his way, I should have to wait until the year 1984, which to me seemed too great a demand! ... But of course I never want to know anything again about Acta Mathematica.

Mittag-Leffler meant this as a kindness but it does show a lack of appreciation of the importance of Cantor's work. The correspondence between Mittag-Leffler and Cantor all but stopped shortly after this event and the flood of new ideas which had led to Cantor's rapid development of set theory over about 12 years seems to have almost stopped.

Philosophy

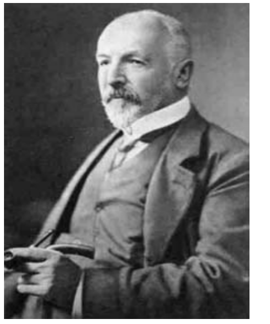
In 1886 Cantor bought a fine new house on Händelstrasse, a street named after the German composer Handel. Before the end of the year a son was born, completing his family of six children. He turned from the mathematical development of set theory towards two new directions, firstly discussing the philosophical aspects of his theory with many philosophers (he published these letters in 1888) and secondly taking over after Clebsch's death his idea of founding the *Deutsche Mathematiker-Vereinigung* which he achieved in 1890. Cantor chaired the first meeting of the Association in Halle in September 1891, and despite the bitter antagonism between himself and Kronecker, Cantor invited Kronecker to address the first meeting.

1893

Kronecker never addressed the meeting, however, since his wife was seriously injured in a climbing accident in the late summer and died shortly afterwards. Cantor was elected president of the Deutsche Mathematiker-Vereinigung at the first meeting and held this post until 1893. He helped to organise the meeting of the Association held in Munich in September 1893, but he took ill again before the meeting and could not attend.

Cantor published a rather strange paper in 1894 which listed the way that all even numbers up to 1000 could be written as the sum of two primes. Since a verification of Goldbach's conjecture up to 10000 had been done 40 years before, it is likely that this strange paper says more about Cantor's state of mind than it does about Goldbach's conjecture.

Beiträge



His last major papers on set theory, *Beiträge zur Begründung der transfiniten Mengenlehre* Part I and Part II appeared in 1895 and 1897, again in *Mathematische Annalen* under Klein's editorship, and are fine surveys of transfinite arithmetic. The rather long gap between the two papers is due to the fact that although Cantor finished writing the second part six months after the first part was published, he hoped to include a proof of the continuum hypothesis in the second part. However, it was not to be.

The Mathematics of the Beiträge

Although these papers can be called surveys, they are really much more than that. Cantor reworks, generalizes and simplifies all of his previous work on set theory and the transfinite numbers.

Cantor starts Part I with his definition of a set :-

By a "set" we mean any collection M into a whole of definite, distinct objects m *(which are called the "elements" of M) of our perception or of our thought.*

He then goes on to revisit the concept of the power of a set which now takes on the characteristics of a transfinite number, similar to, but different from the ordinal numbers.

We call the "power" or "cardinal number" of M that general concept which is derived from the set M by the help of our active powers of thought. The concept is abstracted from the character of the various elements m *and from the order in which they occur in M.*

The Mathematics of the Beiträge

When Cantor finally decided that the transfinite cardinal numbers (the powers of infinite sets) required a separate notation of their own, he felt that all the usual alphabets, the familiar Greek or Roman letters, were too widely used for other purposes. His new numbers deserved something unique. He was always careful about the selection of notation, and it was to be expected that he would make the best choice possible in picking a new symbol for one of the most important concepts of his set theory. Not wishing to invent a new symbol himself, he chose the aleph (א), the first letter of the Hebrew alphabet. The choice was especially clever, as he was happy to admit, since the Hebrew aleph also represents the number 1, and the transfinite numbers, as cardinal numbers, were themselves infinite units. In addition, as the first letter of an alphabet, the aleph could be taken to represent new beginnings, and he certainly believed that his theory represented a new beginning for mathematics.

The Mathematics of the Beiträge

The cardinal number of finite sets is the number of elements in that set, so these are just the natural numbers. The cardinal number of the set $\mathbb{N}$ of all natural numbers, the smallest transfinite cardinal number, is $\aleph_0$.

Cantor goes on to show how to add and multiply cardinal numbers in a manner completely analogous to the methods used for ordinal numbers. Cardinal numbers can also be compared.

If $\mathfrak{a}$ and $\mathfrak{b}$ are cardinal numbers then we define $\mathfrak{a} \leq \mathfrak{b}$ iff there exists a one-to-one function from a set A with cardinal number $\mathfrak{a}$ to a set B with cardinal number $\mathfrak{b}$, and $\mathfrak{a} < \mathfrak{b}$ if $\mathfrak{a} \leq \mathfrak{b}$ and there is no bijection between the sets.

Cantor defines exponentiation of cardinal numbers in general, and as a special case of that, for cardinal number $\mathfrak{a}$, $2^{\mathfrak{a}}$ will be the cardinal number of the power set (the set of all subsets) of a set A with cardinal number $\mathfrak{a}$.

The Mathematics of the Beiträge

This definition leads to one of Cantor's major theorems.

Theorem: For any cardinal number $\mathfrak{a}$, $\mathfrak{a} < 2^{\mathfrak{a}}$.

Pf: For $x \in A$, the function $f: A \to \mathcal{P}(A)$ given by $f(x) = \{x\}$ is an injection, so $\mathfrak{a} \leq 2^{\mathfrak{a}}$.

Now assume that $\mathfrak{a} = 2^{\mathfrak{a}}$ that is, there exists a bijection $g: A \to \mathcal{P}(A)$. Note that $g(x)$ is a subset of A.

Define $S = \{x \in A: x \notin g(x)\}$.

S is a subset of A and since g is onto there must exist some c in A so that $g(c) = S$.

If $c \in S$, then by the definition of S, $c \notin g(c) = S$. $\rightarrow\leftarrow$

If $c \notin S$, then by the definition of S, $c \in g(c)$, so $c \in S$. $\rightarrow\leftarrow$

Our assumption has led to a contradiction, so it must be false.

Thus, $\mathfrak{a} \neq 2^{\mathfrak{a}}$ and we have proved the theorem.

The Mathematics of the Beiträge

This result immediately implies that there are larger and larger cardinal numbers with no possible largest cardinal number!

Having established these larger transfinite cardinal numbers, Cantor looks for a description of them which would shed some light on how they are organized. For this he looks at the ordinal numbers again.

Consider the set of all ordinal numbers which correspond to a set of cardinal number $\aleph_0$, that is a denumerable set. Cantor proves that this set is well-ordered. Therefore, its order type is an ordinal number which can not be in the set. The cardinal number of this set must therefore be greater than $\aleph_0$. He goes on to show that this must in fact be the next largest cardinal number, so he calls it $\aleph_1$. The argument can be repeated producing a next aleph, and then a next and so on.

The Continuum Hypothesis

Returning to his early result that the reals (the continuum) form an uncountable set, the question arises – where does the cardinal number of the reals (denoted by $\mathfrak{c}$) fall in this series of alephs?

By asking the question in this way, Cantor is implicitly assuming that all transfinite cardinal numbers are alephs. This can not be proved without the Axiom of Choice and Cantor had no proof.

For years Cantor believed that the answer to the question was

$$\mathfrak{c} = \aleph_1.$$

But he was never able to prove it which greatly frustrated him. He called this statement ***the continuum hypothesis***.

The Continuum Hypothesis

Cantor felt that he was getting very close to a proof of the continuum hypothesis when he proved that:

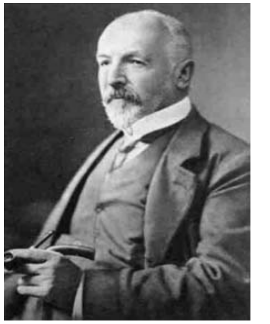
$$\mathfrak{c} = 2^{\aleph_0}.$$

This result led to a more general conjecture, namely that

$$\aleph_{i+1} = 2^{\aleph_i}.$$

If this were true it would (combined with the above result) immediately prove the continuum hypothesis. This conjecture therefore became known as the ***generalized continuum hypothesis***.

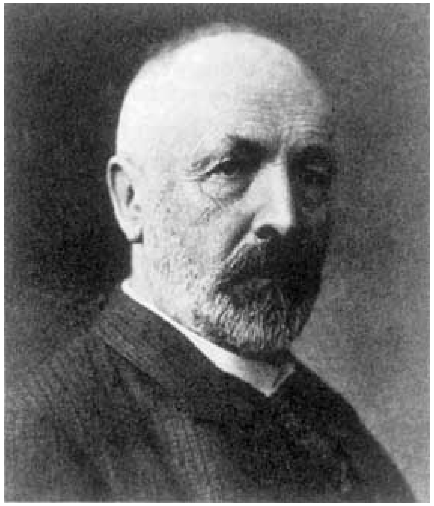
International Congress



In 1897 Cantor attended the first International Congress of Mathematicians in Zurich. In their lectures at the Congress :-

... Hurwitz openly expressed his great admiration of Cantor and proclaimed him as one by whom the theory of functions has been enriched. Jacques Hadamard expressed his opinion that the notions of the theory of sets were known and indispensable instruments.

Paradoxes



At the Congress Cantor met Dedekind and they renewed their friendship. By the time of the Congress, however, Cantor had discovered the first of the paradoxes in the theory of sets. He discovered the paradoxes while working on his survey papers of 1895 and 1897 and he wrote to Hilbert in 1896 explaining the paradox to him. Burali-Forti discovered the paradox independently and published it in 1897. Cantor began a correspondence with Dedekind to try to understand how to solve the problems but recurring bouts of his mental illness forced him to stop writing to Dedekind in 1899.

The Paradox

Cantor realized that a problem had arisen when he considered the set of all ordinal numbers.

As any set of ordinal numbers is well ordered, this set is well ordered and so has an ordinal number (say δ). Since it is the set of all ordinal numbers, d must be in the set. But as the ordinal number of a set is greater than any of the ordinal numbers in the set, we get the contradiction that $\delta < \delta$.

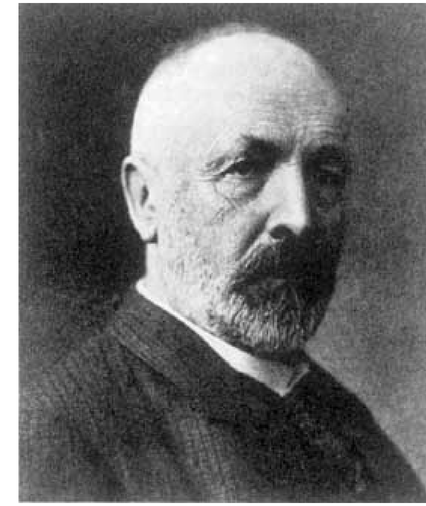
One runs into the same problem if you consider the cardinal number of the set of all cardinal numbers.

Burali-Forti suggested that one way out of the dilemma would be if the set of all ordinal numbers was not well-ordered (this implies that the Well-Ordering Theorem and hence the Axiom of Choice are false). Such a solution would destroy most of the edifice of set theory that Cantor had established.

The Paradox

Cantor's solution to the problem was far less damaging. He called things like the set of all ordinal numbers or the set of all cardinal numbers an ***inconsistent system***. In other words, these objects were not sets (which is the same conclusion that Bertrand Russell reaches when dealing with the set of all sets). However, this now runs counter to Cantor's definition of a set, causing another problem for the general theory.

Cantor was willing to accept a limitation on what constitutes a set, since that would permit him to prove some results that had previously eluded him. However, he wasn't happy about it.



Bacon-Shakespeare

Whenever Cantor suffered from periods of depression he tended to turn away from mathematics and turn towards philosophy and his big literary interest which was a belief that Francis Bacon wrote Shakespeare's plays. For example in his illness of 1884 he had requested that he be allowed to lecture on philosophy instead of mathematics and he had begun his intense study of Elizabethan literature in attempting to prove his Bacon-Shakespeare theory. He began to publish pamphlets on the literary question in 1896 and 1897. Extra stress was put on Cantor with the death of his mother in October 1896 and the death of his younger brother in January 1899.

Waning Years

In October 1899 Cantor applied for, and was granted, leave from teaching for the winter semester of 1899-1900. Then on 16 December 1899 Cantor's youngest son died. From this time on until the end of his life he fought against the mental illness of depression. He did continue to teach but also had to take leave from his teaching for a number of winter semesters, those of 1902-03, 1904-05 and 1907-08. Cantor also spent some time in sanatoria, at the times of the worst attacks of his mental illness, from 1899 onwards. He did continue to work and publish on his Bacon-Shakespeare theory and certainly did not give up mathematics completely. He lectured on the paradoxes of set theory to a meeting of the *Deutsche Mathematiker-Vereinigung* in September 1903 and he attended the International Congress of Mathematicians at Heidelberg in August 1904.

Waning Years

In 1905 Cantor wrote a religious work after returning home from a spell in hospital. He also corresponded with Jourdain on the history of set theory and his religious tract. After taking leave for much of 1909 on the grounds of his ill health he carried out his university duties for 1910 and 1911. It was in that year that he was delighted to receive an invitation from the University of St Andrews in Scotland to attend the 500th anniversary of the founding of the University as a distinguished foreign scholar. The celebrations were 12-15 September 1911 but :-

During the visit he apparently began to behave eccentrically, talking at great length on the Bacon-Shakespeare question; then he travelled down to London for a few days.

Waning Years

Cantor had hoped to meet with Russell who had just published the *Principia Mathematica*. However ill health and the news that his son had taken ill made Cantor return to Germany without seeing Russell. The following year Cantor was awarded the honorary degree of Doctor of Laws by the University of St Andrews but he was too ill to receive the degree in person.

Cantor retired in 1913 and spent his final years ill with little food because of the war conditions in Germany. A major event planned in Halle to mark Cantor's 70 th birthday in 1915 had to be canceled because of the war, but a smaller event was held in his home. In June 1917 he entered a sanatorium for the last time and continually wrote to his wife asking to be allowed to go home. He died of a heart attack.

Cantor

Hilbert described Cantor's work as:-

...the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity.