A Communications Network
A Communications Network

What are the desirable properties of the switching box?
1. Every two users must be connected at a switch.
2. Every switch must "look alike".
3. The fewest number (but bigger than 1) of switches should be used.

To satisfy 2, we would require (at a minimum) that each switch connects the same number of users.

To satisfy 3, we would need to have each two users connected at exactly one switch. It can now be shown that the minimum number of switches is obtained when each pair of switches have a unique user in common.
A Communications Network

Rephrasing these requirements:
1. There exists a switch. (in fact, at least two)
2. Every switch connects exactly n users.
3. Not all users are on the same switch.
4. Each two users are connected at exactly one switch.
5. Each two switches have at least one user in common. (in fact, exactly one)

With the interpretation: Users = Points, Switches = Lines, and Connected = on the above requirements are precisely the axioms of the Fano Geometry (with n = 3), so the switching box design is given by this finite geometry.
More Geometries
Pappus' Theorem

Pappus of Alexandria (340 A.D.)
Pappus' Theorem: If points A, B and C are on one line and A', B' and C' are on another line then the points of intersection of the lines AC' and CA', AB' and BA', and BC' and CB' lie on a common line called the Pappus line of the configuration.
Axioms for the Finite Geometry of Pappus

1. There exists at least one line.
2. Every line has exactly three points.
3. Not all lines are on the same point. [N.B. Change from the text]
4. If a point is not on a given line, then there exists exactly one line on the point that is parallel to the given line.
5. If P is a point not on a line, there exists exactly one point P' on the line such that no line joins P and P'.
6. With the exception in Axiom 5, if P and Q are distinct points, then exactly one line contains both of them.
Theorem 1.10

Each point in the geometry of Pappus lies on exactly three lines.

Pf. Let $X$ be any point. By corrected axiom 3, there is a line not containing $X$. This line contains points $A, B, C$ [Axiom 2]. $X$ lies on lines meeting two of these points, say $B$ and $C$ [Axiom 5]. There is exactly one line through $X$ parallel to $BC$ [Axiom 4].

1. There exists at least one line. 2. Every line has exactly three points. 3. Not all lines are on the same point. 4. If a point is not on a given line, then there exists exactly one line on the point that is parallel to the given line. 5. If $P$ is a point not on a line, there exists exactly one point $P'$ on the line such that no line joins $P$ and $P'$. 6. With the exception in Axiom 5, if $P$ and $Q$ are distinct points, then exactly one line contains both of them.
Theorem 1.10 (cont.)

Each point in the geometry of Pappus lies on exactly three lines.

Pf (cont). There can be no other line through X since by Axiom 4 it would have to meet BC at a point other than A, B or C [Axioms 6 and 5], and this would contradict Axiom 2.

The Pappus geometry has 9 points and 9 lines.

1. There exists at least one line. 2. Every line has exactly three points.
3. Not all lines are on the same point.
4. If a point is not on a given line, then there exists exactly one line on the point that is parallel to the given line.
5. If P is a point not on a line, there exists exactly one point P' on the line such that no line joins P and P'.
6. With the exception in Axiom 5, if P and Q are distinct points, then exactly one line contains both of them.
Girard Desargues
(1591 - 1661)

Brouillon project d'une atteinte aux evenemens des recontres du Cone avec un Plan

(Rough draft for an essay on the results of taking plane sections of a cone) 1639

Father of Projective Geometry
Desargues' Theorem

Triangles perspective from a point.
Desargues' Theorem

Triangles perspective from a line.
Desargues' Theorem

Two triangles perspective from a point are perspective from a line.
Desargues' Theorem

Desargues' Theorem: In a projective plane, two triangles are said to be *perspective from a point* if the three lines joining corresponding vertices of the triangles meet at a common point called the *center*. Two triangles are said to be *perspective from a line* if the three points of intersection of corresponding lines all lie on a common line, called the *axis*. Desargues' theorem states that two triangles are perspective from a point if and only if they are perspective from a line.

Desargues' Configuration has 10 points and 10 lines.
A new axiom system H*

1. Two distinct points are on exactly one line.
2. Not all points are on the same line.
3. For every point and line not through that point, there are \( n \) lines \((n > 1)\) on the point which do not meet the line.
4. There exists a line containing \( m > 0 \) points.

Any system of points and lines that satisfies this set of axioms is called an \(<m,n>\)-hyperbolic plane.
Example

Take 5 points in Euclidean 3-space with the property that no more than two points lie on a line and no more than three points lie on a common plane.

This is a $<2,2>$-hyperbolic plane.
Problems and Questions

1. Show that you can assume that \( m \geq 2 \).

2. Show that all lines contain \( m \) points.

3. What is the total number of points in an \(<m,n>-hyperbolic plane?\)

4. What is the total number of lines?

5. Is there a numerical relationship between \( m \) and \( n \)? If so, what is it?

6. If \( m > 2 \), are there any examples of \(<m,2>-hyperbolic planes?\)
Plane Duality Again

Three Point Geometry:

1. There exist exactly 3 points in this geometry.
2. Two distinct points are on exactly one line.
3. Not all the points of the geometry are on the same line.
4. Two distinct lines are on at least one point.

**Theorem 1.1**: Two distinct lines are on exactly one point.

**Theorem 1.2**: The three point geometry has exactly three lines.

Three Line Geometry:

1. There exist exactly 3 lines in this geometry.
2. Two distinct lines are on exactly one point.
3. Not all the lines of the geometry are on the same point.
4. Two distinct points are on at least one line.

**Theorem**: Two distinct points are on exactly one line.

**Theorem**: The three line geometry has exactly three points.
Plane Duals Again

We can show that not all of the points of the three line geometry are on the same line – (hmmm, that's a good quiz question)

That means that all the axioms of the three point geometry are true statements (either axioms or theorems) in the three line geometry. So, the three line geometry is a three point geometry.

The converse also holds, that is, all the axioms of the three line geometry are true statements in the three point geometry, so the three point geometry is the three line geometry.

When a geometry is the same as its plane dual geometry we say that the geometry is self-dual.
Self-Dual Geometries

Other geometries that we have looked at which are self-dual are:

1. Fano Geometry
2. Desargues Geometry

It should be clear that in order for a geometry to possibly be self-dual, it is necessary (but not sufficient) that the number of points equals the number of lines. So, we can say without any work that the four line geometry, the four point geometry and Young's geometry can not be self-dual.

What about the Pappus Geometry? (9 points/9 lines)
Fano's Geometry

1. There exists at least one line.
2. Every line of the geometry has exactly 3 points on it.
3. Not all points of the geometry are on the same line.
4. For two distinct points, there exists exactly one line on both of them.
5. Each two lines have at least one point on both of them.

Theorem 1.7: Each two lines have exactly one point in common.

Theorem 1.8 : Fano's geometry consists of exactly seven points and seven lines.
The Fano geometry is the smallest member of an infinite family of self-dual geometries that are denoted by $\text{PG}(n,q)$. This stands for "the projective geometry of dimension $n$ and order $q". Here $n$ may be any positive integer, but $q$ must be a power of a prime number. The axioms that define $\text{PG}(2,q)$ are the same as those for the Fano geometry, except that "3 points on a line" is replaced by "$q+1$ points on a line". Fano's geometry is thus $\text{PG}(2,2)$.

Note that the definition of order in the text is incorrect. It should be $q$, not $q+1$ as stated on pg. 31.
The number of points (and also the number of lines, since these are self-dual) in a PG(n,q) is given by:

\[ \frac{q^{n+1} - 1}{q - 1} \]

So, PG(2,2) has 7 = (8-1)/(2-1) points and PG(2,3) has 13 = (27-1)/(3-1) points.

These geometries are studied in detail in Math 4210.
Euclidean Geometry

Euclid's Postulates

- A straight line can be drawn from any point to any point.
- A finite straight line can be produced continuously in a straight line.
- A circle may be described with any point as center and any distance as radius.
- All right angles are equal to one another.

- If a transversal falls on two lines in such a way that the interior angles on one side of the transversal are less than two right angles, then the lines meet on that side on which the angles are less than two right angles.
Euclidean Geometry

Euclid's unstated assumptions.

1. Lack of continuity.
2. Pasch's Axiom.
3. Order of points and betweeness.
4. Problems with superposition.

However, even with these problems, Euclid did not ever write down a false statement.

Modern treatments have either corrected or provided alternatives to Euclid's scheme.
Pasch's Axiom

If a line intersects one side of a triangle, not at a vertex, then it must intersect another side of the triangle.
Modern Treatments

Moritz Pasch (1882)
Guiseppi Peano (1889)
David Hilbert (1902) - in the style of Euclid
Oswald Veblen (1904, 1911)
E.V. Huntington (1913)
Henry Forder (1927)
G.D. Birkhoff (1932) - a different approach
Fixing up Euclid

David Hilbert (1862 - 1943)

Undefined terms: Point, Line, Plane, Between, Congruent, On.

20 Axioms needed for Euclidean Geometry.

III. In a plane $\alpha$, there can be drawn through any point A, lying outside of a straight line $a$, one and only one straight line that does not intersect the line $a$. 
An Alternative to Euclid

G. D. Birkhoff (1884 - 1944)

I. The points of any line can be put into one-to-one correspondence with the real numbers.

II. Two distinct points determine a unique line.

III. The half-lines through any point can be put into one-to-one correspondence with the real numbers (mod $2\pi$).

IV. There exist similar, but not congruent, triangles.
Axiom Systems

Desirable properties of Axiom Systems

1. consistent – should not contain contradictions.
2. complete – all theorems can be derived from them.
3. independent – no axiom can be proved from the others.

*Note:* We show independence by use of models.

While independence is mathematically desirable, it is often not pedagogically useful.
Desargues Geometry

Local Definitions for this geometry only!

The line $l$ is a **polar** of the point $P$ if there is no line connecting $P$ and a point on $l$.

B'C' is a polar of A

The axis is a polar of the center.
Desargues Geometry

Local Definitions for this geometry only!

The point P is a **pole** of the line l if there is no point common to l and any line on P.

B is a pole of A'C'

The center is a pole of the axis.
Axioms for Desargues' Geometry

1. There exists at least one point.
2. Each point has at least one polar.
3. Every line has at most one pole.
4. Two distinct points are on at most one line.
5. Every line has exactly three distinct points on it.
6. If a line does not contain a point P, then there is a point on both the line and any polar of P.
A Few Theorems

**Proposition.** P is the pole of \( l \) if and only if \( l \) is a polar of P.

**Proposition.** If P is on a polar of Q then Q is on every polar of P.

**Theorem 1.11** Every line in the geometry of Desargues has exactly one pole.

**Theorem 1.12** Every point in the geometry of Desargues has exactly one polar.
A Proposition

P is the pole of $l$ iff $l$ is a polar of P.

\textit{Pf}: Suppose that the point P is the pole of line $l$ and BWOC that $l$ is not a polar of P. Since $l$ is not a polar of P, there is a line, say $w$, which connects P with a point, say Q, on $l$. But then $w$ is a line on P which has the point Q in common with $l$, which contradicts the definition of P being the pole of $l$. So, we may conclude that $l$ is a polar of P.

Now suppose that $l$ is a polar of P and P is not the pole of $l$. Since P is not the pole of $l$, there is a line $s$ on P which has a point R in common with $l$. But this contradicts the definition of $l$ being a polar of P.

The line $l$ is a \textit{polar} of the point P if there is no line connecting P and a point on $l$.

The point P is a \textit{pole} of the line $l$ if there is no point common to $l$ and any line on P.
A Proposition
If P is on a polar of Q then Q is on every polar of P.

Pf: Let P be on a polar of Q, say \( m \), and assume that Q is not on some polar of P. Let \( l \) be this polar of P. Since Q is not on \( l \), Axiom 6 says that there is a point \( R \) on \( l \) and on \( m \). But now \( m \) is a line on P which has a point in common with a polar of P, which contradicts the definition of \( l \) being a polar of P. Therefore, Q must be on this polar of P.
Theorem 1.11
Every line in the geometry of Desargues has exactly one pole.

Pf: By Axiom 3, every line has at most one pole, so we need to show that every line has at least one pole.

Let \( l \) be any line. Let \( R \) and \( S \) be two points on \( l \) (these exist by Axiom 5). Let \( s \) be a polar of \( S \) (exists by Axiom 2). \( R \) can not be on \( s \), for if it was \( S \) would be connected to a point (\( R \)) on its polar by the line \( l \). So, by Axiom 6, \( s \) and any polar \( r \) of \( R \) have a point in common, say \( P \). Since \( P \) is on \( s \) and \( r \), \( S \) and \( R \) are on any polar of \( P \). By Axiom 4, \( S \) and \( R \) can only be on one line, so \( l \) must be a polar of \( P \). By our first proposition, \( P \) is thus a pole of \( l \).

1. There exists at least one point.
2. Each point has at least one polar.
3. Every line has at most one pole.
4. Two distinct points are on at most one line.
5. Every line has exactly three distinct points on it.
6. If a line does not contain a point \( P \), then there is a point on both the line and any polar of \( P \).
Theorem 1.12

Every point in the geometry of Desargues has exactly one polar.

Pf: By Axiom 2, each point has at least one polar, so we must show that no point can have more than one polar.

Assume that point P has polars $p_1$ and $p_2$. Since P is not on $p_1$, Axiom 6 says that $p_1$ and $p_2$ meet at a point Q. Now, let R be a point of $p_1$, not equal to Q (exists by Axiom 5). Since R is on a polar of P, P is on every polar of R (which exist by Axiom 2). In turn, P is on a polar of R, so R is on every polar of P. Thus, R is on $p_2$, and since Q and R are on both polars, $p_1 = p_2$ by Axiom 4.

1. There exists at least one point.
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6. If a line does not contain a point P, then there is a point on both the line and any polar of P.