Non-Euclidean Geometry

The Parallel Postulate

Non-Euclidean Geometry is **not** not Euclidean Geometry. The term is usually applied only to the special geometries that are obtained by negating the parallel postulate but keeping the other axioms of Euclidean Geometry (in a complete system such as Hilbert's).

History of the Parallel Postulate

Saccheri (1667-1733) "***Euclid Freed of Every Flaw***" (1733, published posthumously)

The first serious attempt to prove Euclid's parallel postulate by contradiction. This Jesuit priest succeeded in proving a number of interesting results in hyperbolic geometry, but reached a flawed conclusion at the end of the work.

Lambert (1728 - 1777) "***Theory of Parallels***" (also published posthumously)

Similar in nature to Saccheri's work, and probably influenced by it. However, Lambert was astute enough to realize that he had not proved the parallel postulate. He did not publish this work himself.

History of the Parallel Postulate

Nikolai Ivanovich Lobachewsky (1793-1856) "***On the Principles of Geometry***" (1829)
The first published account of hyperbolic geometry, in Russian. Lobachewsky developed his ideas from an analytical (trigonometric) viewpoint.

Johann ('Janos') Bolyai (1802-1860) "***Appendix exhibiting the absolute science of space: independent of the truth or falsity of Euclid's Axiom XI (by no means previously decided)***" in Wolfgang Bolyai's, *Essay for studious youths on the elements of mathematics* (1832)
An approach similar to Lobachewsky's, but he was unaware of Lobachewsky's work.

History of the Parallel Postulate

Gauss (1777-1855) - unpublished.

In a letter to Wolfgang Bolyai, after receiving a copy of Janos' appendix, Gauss wrote:

"If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it, would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years. So I remained quite stupefied. So far as my own work is concerned, of which up till now I have put little on paper, my intention was not to let it be published during my lifetime. ... It is therefore a pleasant surprise for me that I am spared this trouble, and I am very glad that it is just the son of my old friend, who takes the precedence of me in such a remarkable manner."

History of the Parallel Postulate

Later, in a letter to Bessel in 1829, Gauss wrote:
"It may take very long before I make public my investigations on this issue: in fact, this may not happen in my lifetime for I fear the 'clamor of the Boeotians.' "

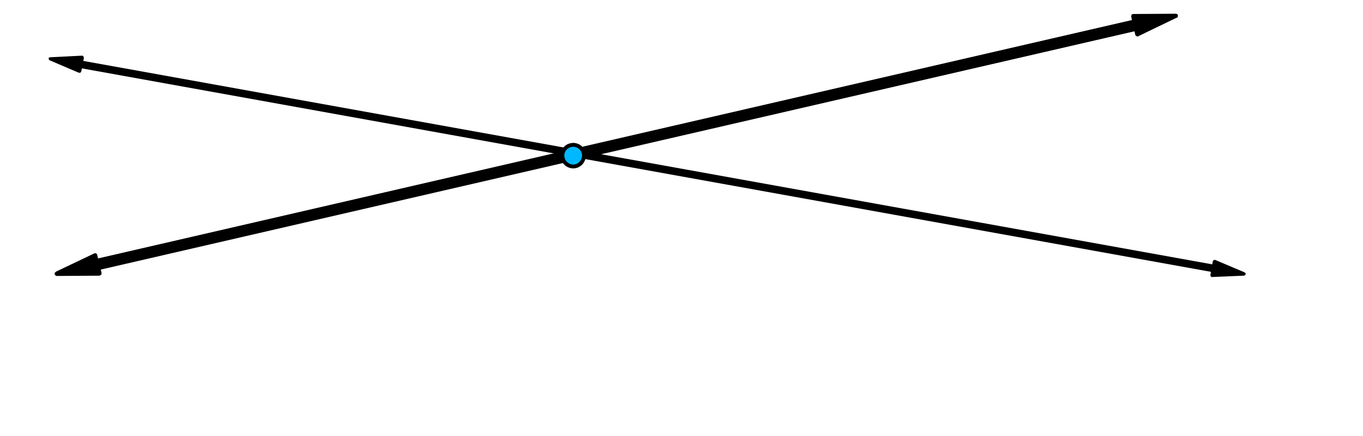
Boeotia was a province of ancient Greece whose inhabitants were known for their dullness and ignorance.

Riemann (1826-1866) "***On the Hypotheses Which Lie at the Foundation of Geometry***" (1854)
In this inaugural address Riemann outlined the basic ideas underlying Elliptic Geometry. It was the simplest example of what are now called Riemannian geometries.

Hyperbolic Geometry

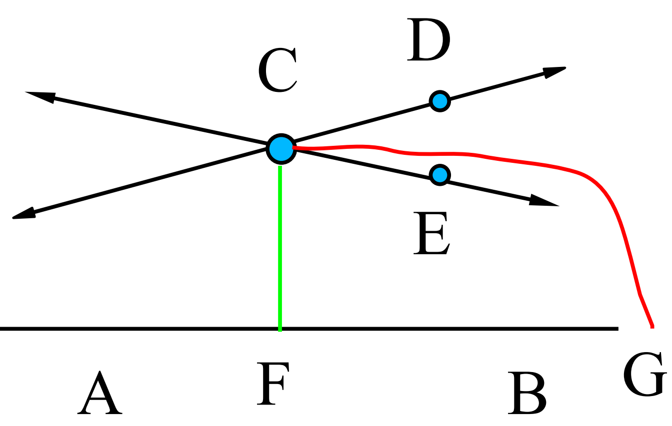
Characteristic Postulate

Through a given point C, not on a given line AB, passes more than one line in the plane not intersecting the given line.



Theorem 9.1

Through a given point C, not on a given line AB, pass an infinite number of lines not intersecting the given line.



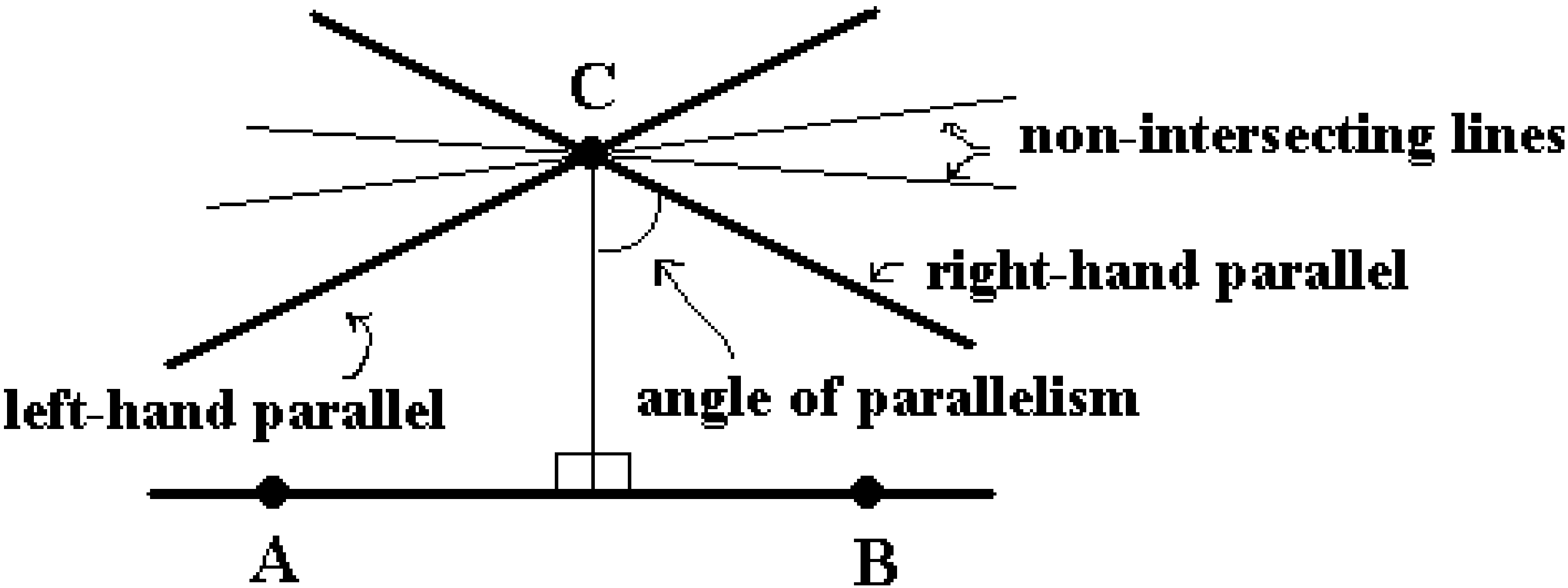
Pf: There are an infinite number of lines through C in the interior of the angle DCE. Suppose that one of these lines meets AB at a point G.

Drop the perpendicular from C to AB, meeting AB at F. The line CE enters triangle CFG and by Pasch's Axiom it must meet side FG (= AB) →←. Thus, the line does not meet AB.

Non-intersecting lines

Def: Given a point C not on a line AB, the first line through C in either direction that does not meet AB is called a ***parallel line***. Other lines through C which do not meet AB are called ***nonintersecting lines***. The two parallel lines through C are called the ***right-hand parallel*** and ***left-hand parallel***. The angle determined by the line from C perpendicular to AB and either the right or left hand parallel is called the ***angle of parallelism***.

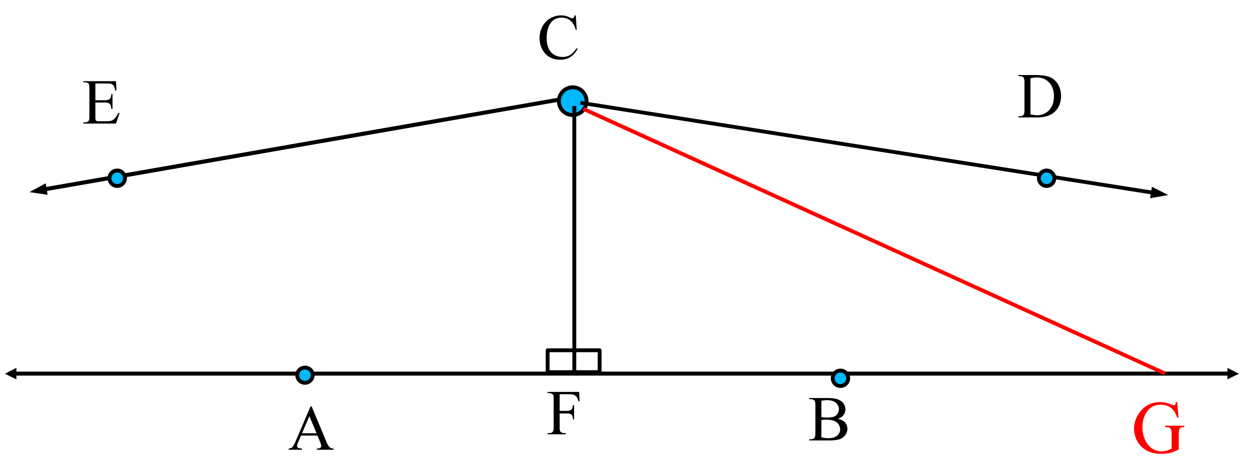
Non-intersecting lines



Note: While it is true that there is a first line through C which does not meet AB, there is no last line through C which does meet AB.

Theorem 9.2

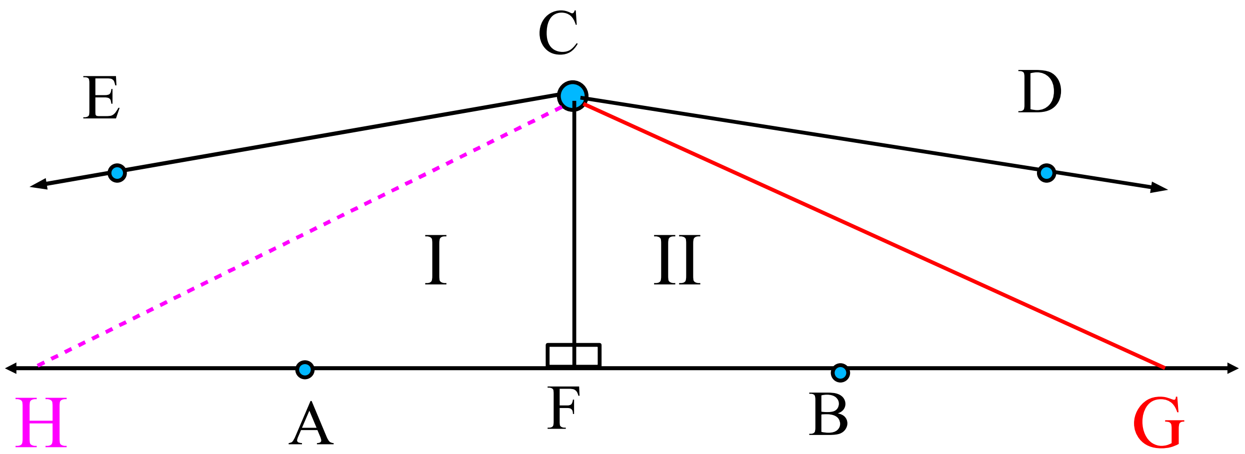
The two angles of parallelism for the same distance are congruent and acute.



Pf: Suppose that ∠FCE and ∠FCD are the angles of parallelism for CF, but are not congruent. WLOG we may assume ∠FCD is the larger angle. Since CD is the right-hand parallel, there exists a point G on AB so that ∠FCG is congruent to ∠FCE.

Theorem 9.2

The two angles of parallelism for the same distance are congruent and acute.



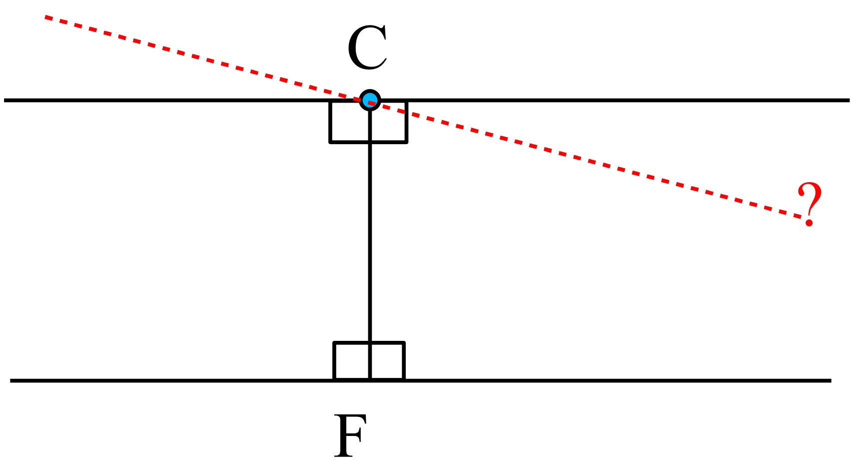
Construct point H on AB so that FH = FG.

ΔI ≅ ΔII by SAS. Thus, ∠FCH ≅ ∠FCG ≅ ∠FCE and so, CH = CE but this contradicts the fact that CE is the left-hand parallel. Therefore, the two angles of parallelism must be congruent.

Theorem 9.2

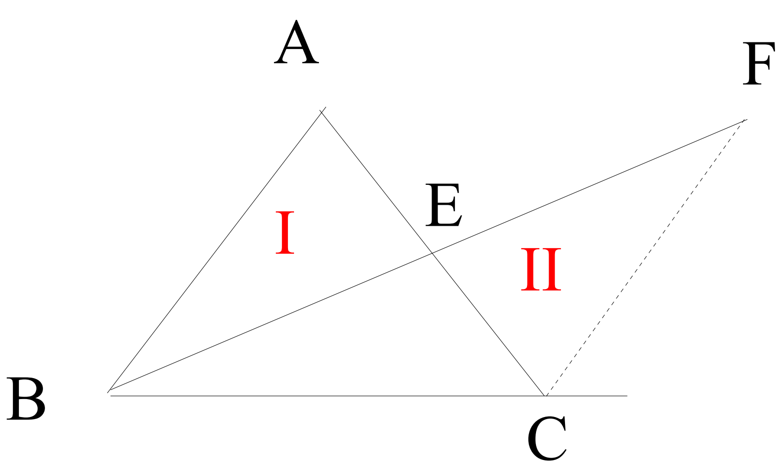
The two angles of parallelism for the same distance are congruent and acute.

The angles of parallelism can not be right angles, because any other line through C would have to (on one side or the other) lie below the parallel line, and thus meet line AB. This contradicts the characteristic postulate.



Exterior Angles

The well known result that the measure of an exterior angle of a triangle is the sum of the measures of the opposite interior angles, requires the use of the parallel postulate. However, Euclid proved a weaker result (Prop. 16) which does not use the parallel postulate, namely, the exterior angle of a triangle is greater than either of the opposite interior angles.



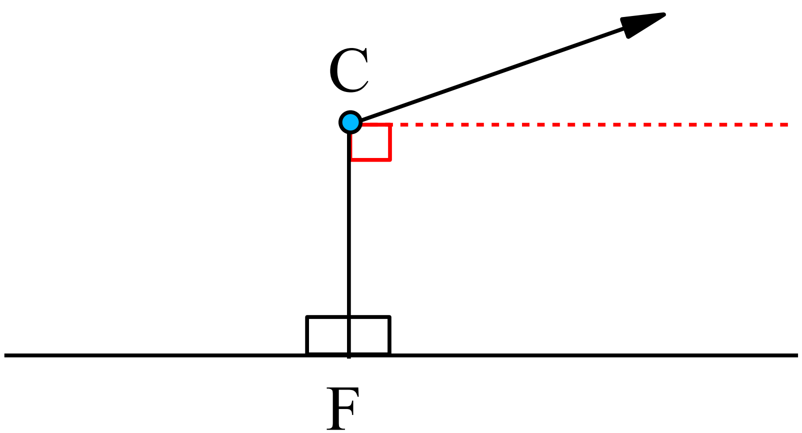
Extend the median BE so that BE = EF and join F and C.

Triangles I and II are congruent by SAS, so ∠BAE ≅ ∠ECF. But the exterior angle at C is greater than ∠ECF, giving our conclusion. The angle at B is dealt with similarly.

Theorem 9.2

The two angles of parallelism for the same distance are congruent and acute.

The angles of parallelism can not be obtuse since then the perpendicular drawn to CF at C would be a non-intersecting line below the parallel line at C.



Since the angle of parallelism can not be a right angle or an obtuse angle, it must be an acute angle.

Omega Triangles

Def: All the lines that are parallel to a given line in the same direction are said to intersect in an ***omega point*** (***ideal point***).

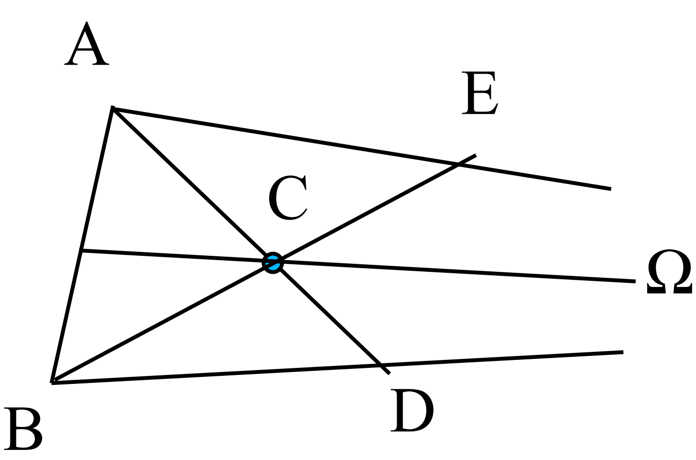
Def: The three sided figure formed by two parallel lines and a line segment meeting both is called an ***Omega triangle***.

Omega Triangle

Theorem 9.3

The axiom of Pasch holds for an omega triangle, whether the line enters at a vertex or at a point not a vertex.

Pf: Let C be any interior point of the omega triangle ABΩ. We first examine lines which enter the omega triangle at a "vertex".



Line AC intersects BΩ since AΩ is the first non-intersecting line to BΩ.

Line BC intersects AΩ since BΩ is the first non-intersecting line to AΩ.

A line CΩ entering the omega triangle at Ω meets AB by applying Pasch's Axiom to the ordinary triangle ABC.

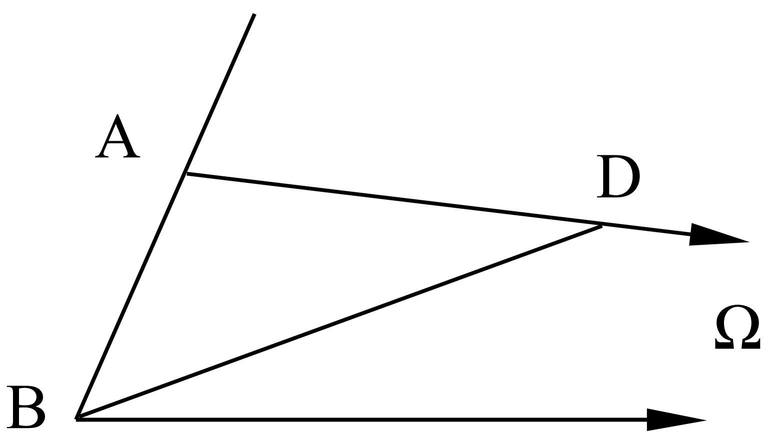
Exterior Angles

Since the exterior angle theorem is proved without using the parallel postulate, it remains true in hyperbolic geometry. What is surprising is that it is also true for omega triangles (which are of course not triangles)!!

Theorem 9.4

For any omega triangle ABΩ, the measures of the exterior angles formed by extending AB are greater than the measures of their opposite interior angles.

Pf: We will prove this result by eliminating the other possibilities. We first assume that the exterior angle at A is smaller than the interior angle at B of the omega triangle ABΩ.



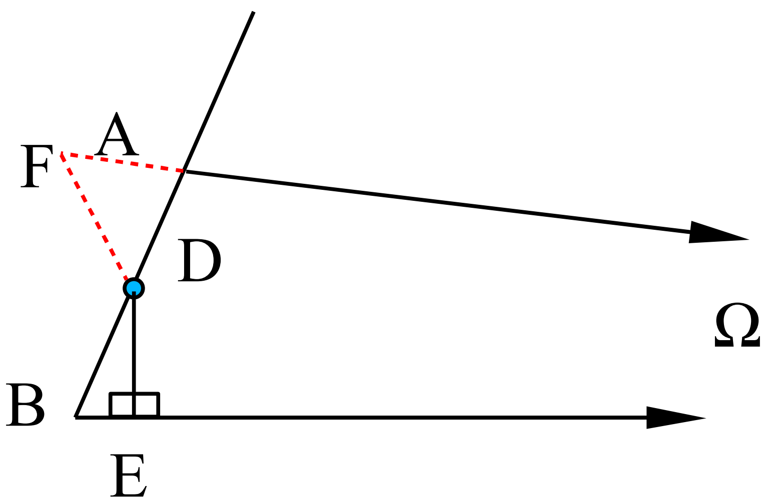
Since the angle at A is smaller than the angle at B, we can find a point D on AΩ so that the exterior angle at A ≅ ∠ABD.

This is a contradiction since ABD is an ordinary triangle.

Theorem 9.4

For any omega triangle ABΩ, the measures of the exterior angles formed by extending AB are greater than the measures of their opposite interior angles.

We now assume that the exterior angle at A is equal to the interior angle at B of the omega triangle ABΩ.



Let D be the midpoint of AB and drop the perpendicular from D to BΩ.

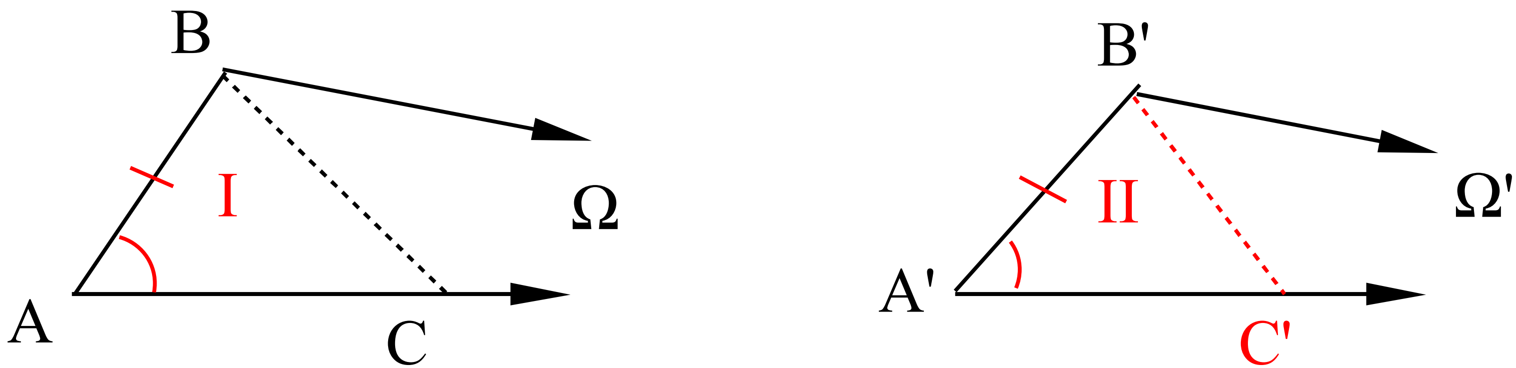
Extend AΩ so that AF = BE.

Triangles DBE and AFD are congruent by SAS and so EF is a straight line and perpendicular to AΩ.

This is a contradiction since the angle of parallelism for EF is acute.

Theorem 9.5

Omega triangles ABΩ and A'B'Ω' are congruent if the sides of finite length are congruent and if a pair of corresponding angles at A and A' or B and B' are congruent.



Assume that AB = A'B' and ∠A ≅ ∠A'.

WLOG assume that ∠B > ∠B'. We can find a point C on AΩ so that ∠ABC ≅ ∠B'.

Locate C' on A'Ω' so that A'C' = AC.

ΔI ≅ ΔII (SAS) which gives the contradiction that ∠B' ≅ ∠A'B'C'.

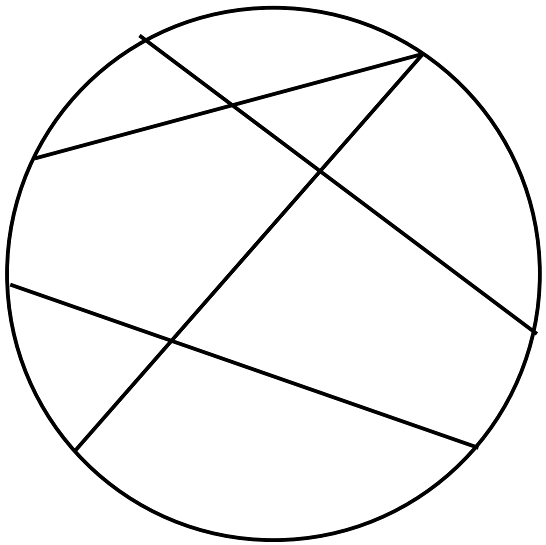
Theorem 9.6

Omega triangles $AB\Omega$ *and* $A'B'\Omega'$ *are congruent if the pair of angles at* A *and* A' *are congruent and the pair of angles at* B *and* B' *are congruent.*

Proof omitted.

Models for Hyperbolic Geometry

Klein (1849-1925) model for hyperbolic geometry.

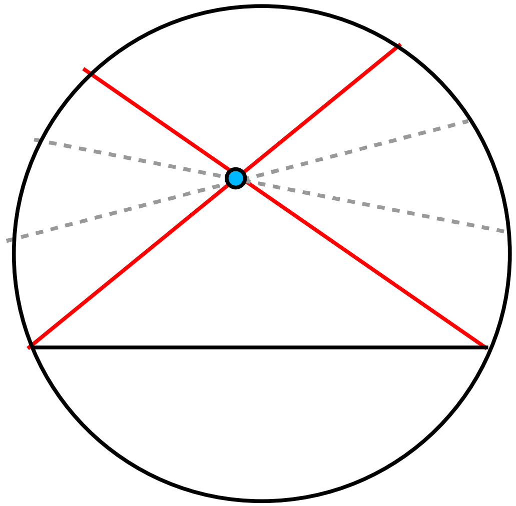


POINTS: The interior points of a fixed circle.

LINES: The chords of this circle.

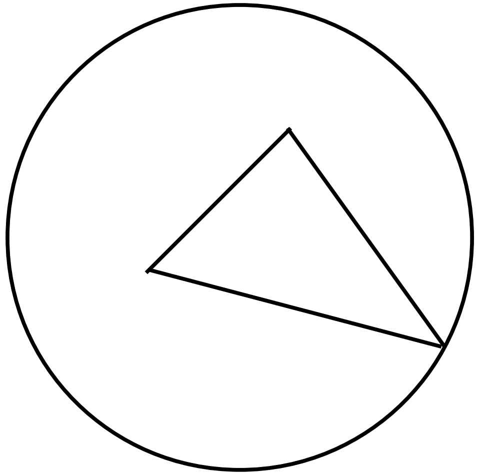
Omega (ideal) points are the points on the circle (which are not in the geometry). Thus, parallel lines are those which meet on the circle.

The Klein Model



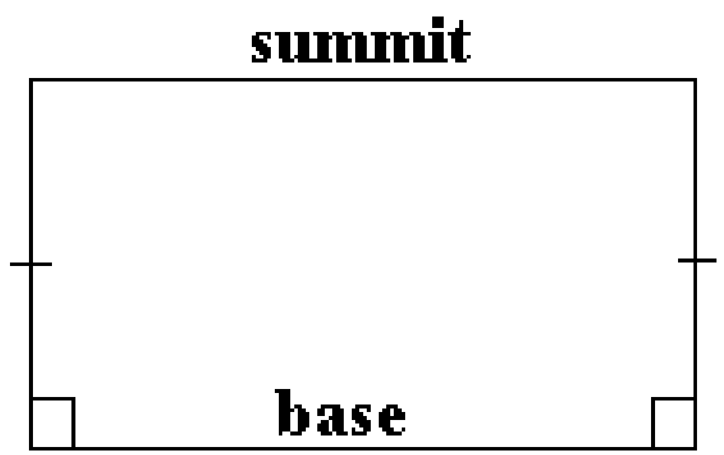
The characteristic postulate:

An omega triangle:



Saccheri Quadrilaterals

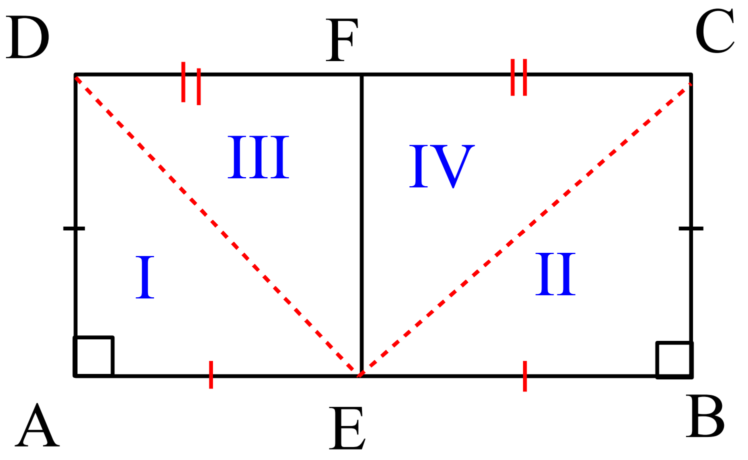
Def: ***Saccheri quadrilateral***: A quadrilateral with two opposite sides of equal length, both perpendicular to a third side. The side of the quadrilateral which makes right angles with both the equal length sides is called the ***base***, and the fourth side is called the ***summit***.



Saccheri Quadrilateral

Theorem 9.7

The segment joining the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both.



Pf: Draw the lines DE and EC.

Triangles I and II are congruent by SAS. Therefore, DE = CE.

Triangles III and IV are now congruent by SSS.

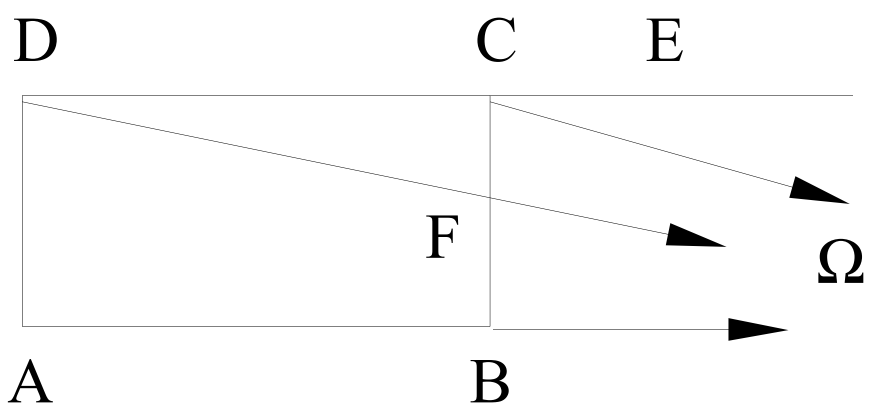
Angles DFE and CFE are congruent and supplementary ... 90^o each.

Angles AEF and BEF are the sums of two congruent angles and so they are also congruent and supplementary.

Theorem 9.8

The summit angles of a Saccheri quadrilateral are congruent and acute.

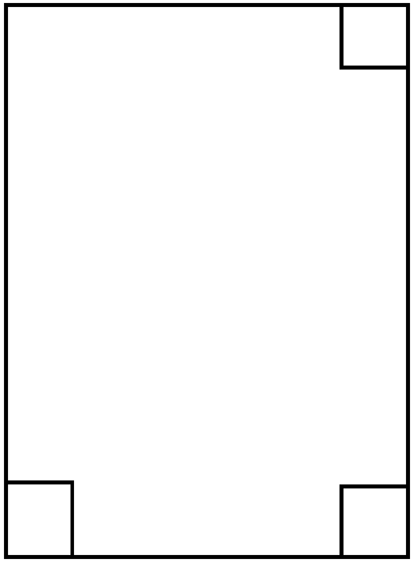
Pf: The congruence of the summit angles follows from the congruence of triangles I and II, and III and IV in the diagram of the last theorem.



Draw the right parallels to AB at D and C. The angles of parallelism are equal since AD = BC. $\angle EC\Omega$ is greater than $\angle CDF$ since it is an exterior angle of an omega triangle. Thus, $\angle ECB > \angle CDA$. But $\angle DCB \cong \angle CDA$, so $\angle DCB$ is acute.

Lambert Quadrilaterals

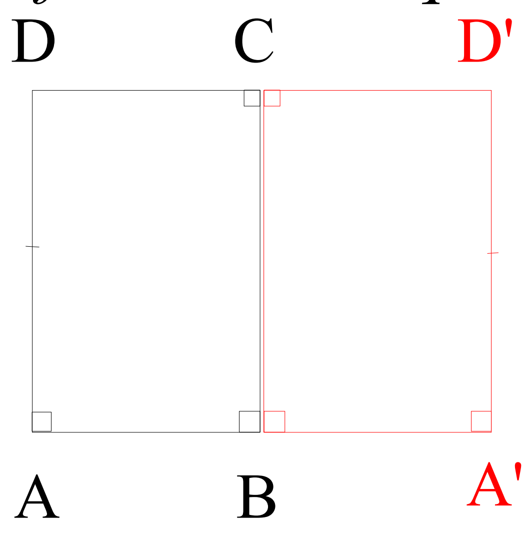
Def: ***Lambert quadrilateral***: A quadrilateral in which three of the angles are right angles.



Lambert Quadrilateral

Theorem 9.9

The fourth angle of a Lambert quadrilateral is acute.

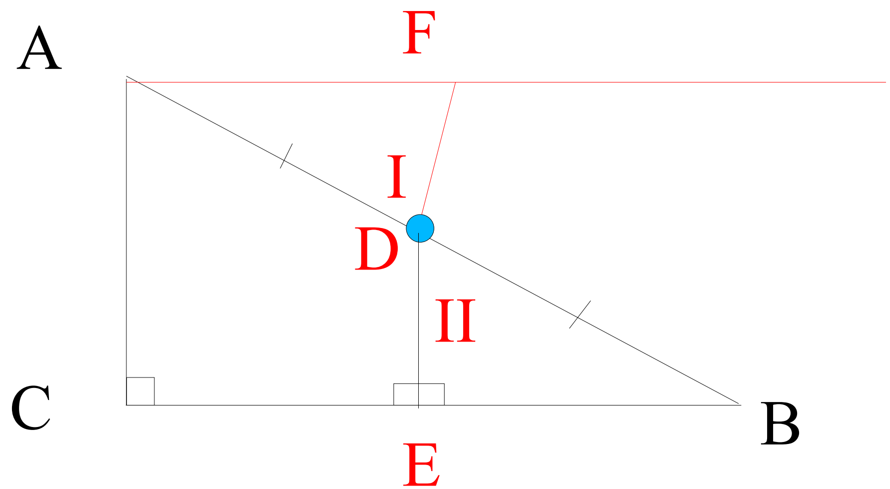


Pf: Duplicate the Lambert quadrilateral on the other side of BC.

ADD'A' is a Saccheri quadrilateral and by Theorem 9.8 the angle at D is an acute angle.

Angle Measure

Theorem 9.10: The sum of the measures of the angles of a right triangle is less than π .

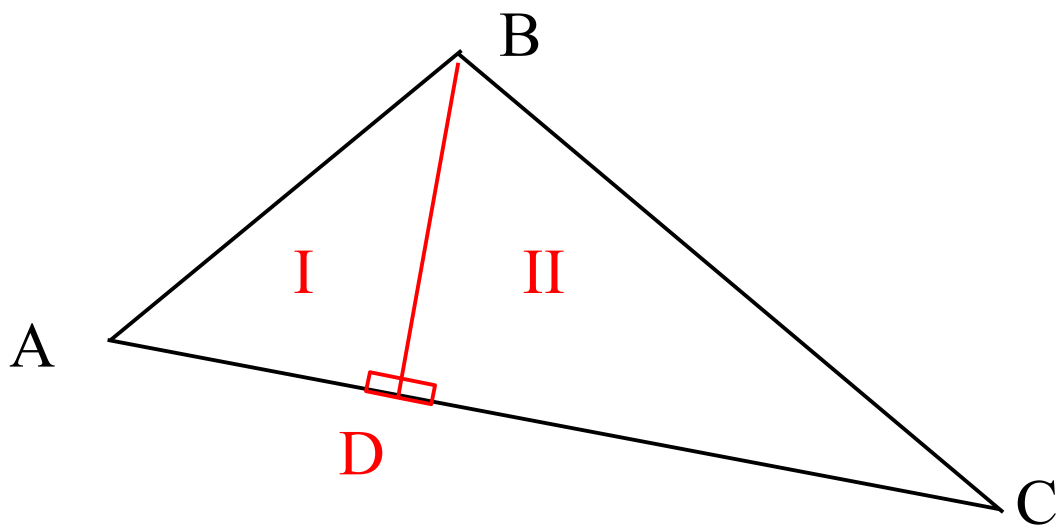


Pf: Drop a perpendicular from D, the midpoint of hypotenuse AB to side CB.

Construct $\angle FAD \cong \angle EBD$ with $FA = EB$. $\Delta I \cong \Delta II$ by SAS. $\angle AFD$ is a right angle, FE is a straight line, ACEF is a Lambert quadrilateral. The angle sum of ΔABC is 90^{o} + an acute angle

Angle Measure

Theorem 9.11: The sum of the measures of the angles of any triangle is less than π.



Pf: Drop the perpendicular from B to AC.

By Theorem 9.10, the angle sums of triangles I and II are both less than π. Thus, the angle sum of triangle ABC is $< 2\pi - \pi$ (angles at D).

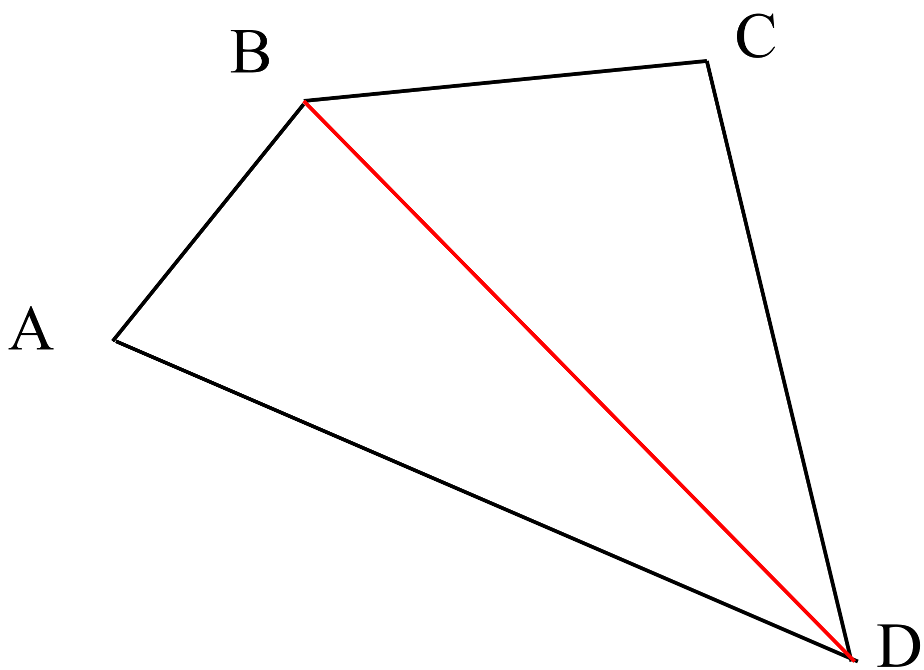
Defect of a Triangle

Def: The ***defect*** of a triangle is the difference between π and the sum of the angles of the triangle.

Theorem 9.11 says that in hyperbolic geometry, every triangle has positive defect. In Euclidean geometry, every triangle has defect zero.

More Angle Measures

Theorem 9.12: The sum of the measures of the angles of any convex quadrilateral is less than 2π.

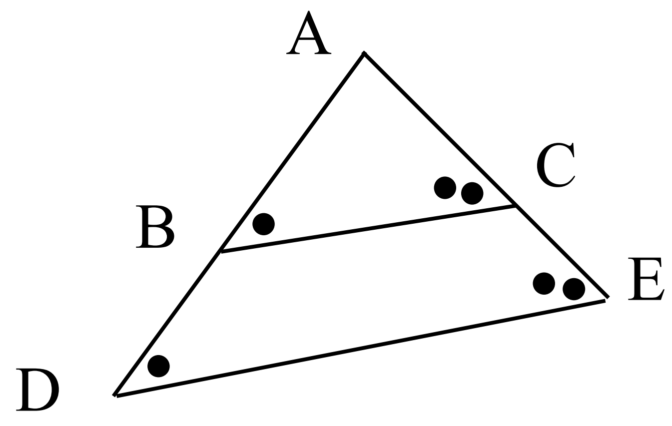


More Angle Measures

Theorem 9.13: Two triangles are congruent if the three pairs of corresponding angles are congruent.

Pf: Suppose that triangles ABC and A'B'C' have corresponding angles congruent. BWOC we will assume that AB ≠ A'B' and WLOG we may assume that A'B' > AB. Two cases now arise, either A'C' > AC or A'C' ≤ AC. In either case, we extend (if necessary) the sides AB and AC of triangle ABC and find points D on AB and E on AC, so that AD = A'B' and AE = A'C'.

In the first case we have:



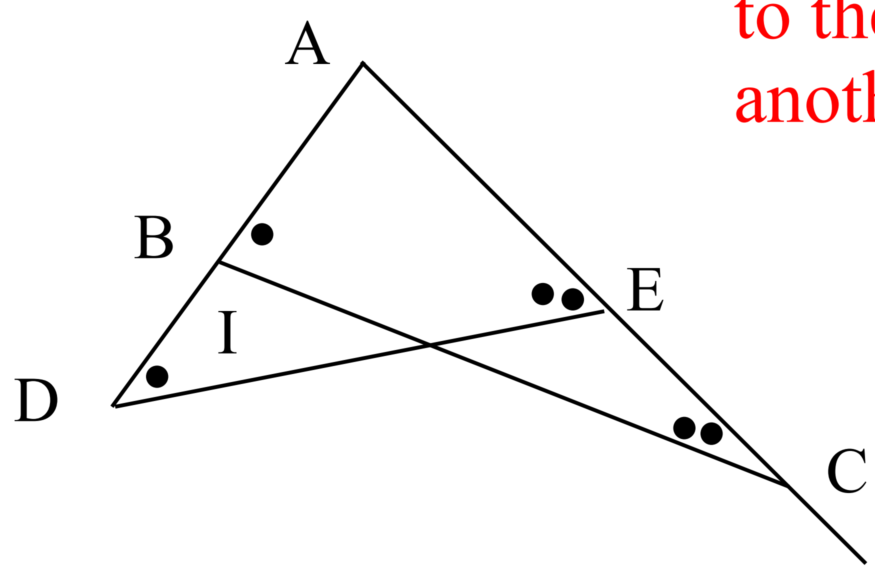
The angle sum of convex quadrilateral BCED is 360°, a contradiction.

More Angle Measures

Theorem 9.13: Two triangles are congruent if the three pairs of corresponding angles are congruent.

In the second case we have:

The exterior angle at B of triangle I is congruent to the interior angle at D, another contradiction.



Thus, AB = AD = A'B'. And we have that triangles ABC and A'B'C' are congruent by ASA.

Gamma Points

Def: Two non-intersecting lines are said to meet at a ***gamma point*** (***ultra-ideal point***).

Theorem 9.14: *Two non-intersecting lines have a common perpendicular*.

The idea behind this proof is to find a Saccerhi quadrilateral which has the two non-intersecting lines as base and summit. Then the line joining the midpoints of the base and summit is perpendicular to both.

Area

The concept of area in Euclidean geometry is based on the arbitrary standard that the area of a unit square is one.

Since rectangles, and therefore squares, do not exist in hyperbolic geometry, the concept of area must be defined in a different way than that which is used in Euclidean geometry.

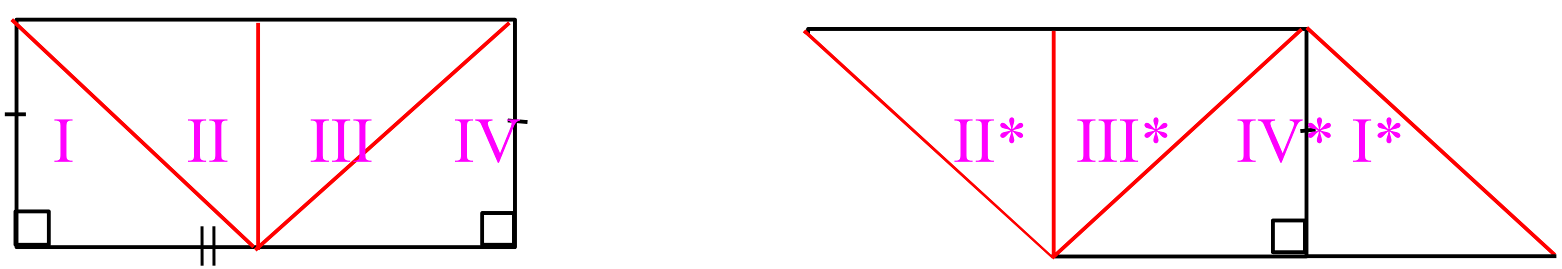
We will now examine an idea upon which we can base a concept of area in hyperbolic geometry.

Equivalence

Def: Two polygons are called ***equivalent*** if they can be partitioned into the same finite number of pairs of congruent triangles.

Congruent polygons are equivalent:

But, non-congruent polygons can also be equivalent:



Equivalence

Theorem 9.15: *Two triangles are equivalent iff they have the same defect*.

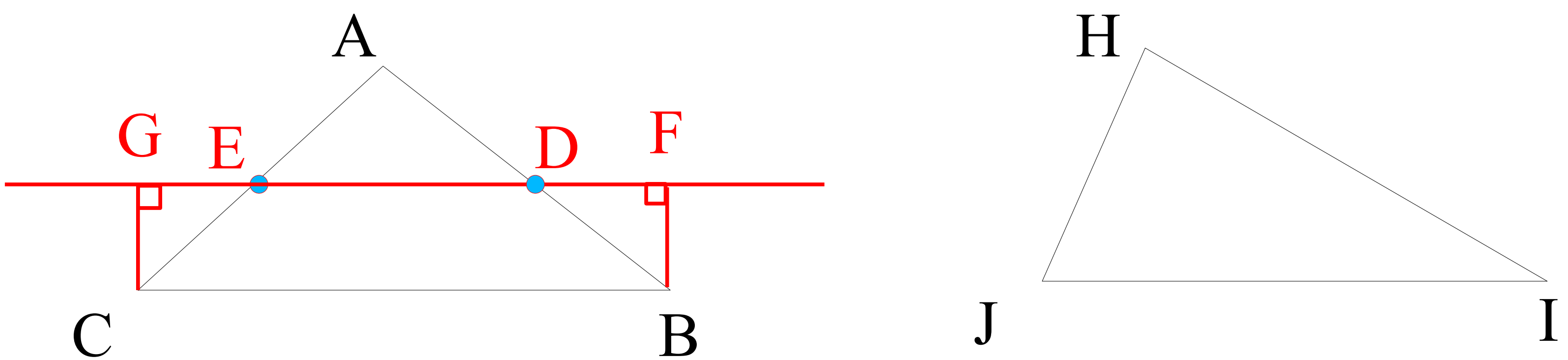
Pf: By definition if two triangles are equivalent, they can be partitioned into a finite number of pairs of congruent triangles. The defect of each triangle of the original pair is the sum of the defects of the triangles that partition it, thus these two triangles would have the same defect.

We now turn to the more difficult direction of this proof and show that two triangles with the same defect are in fact equivalent.

We will look at a special case first. This is the case where the two triangles have a pair of congruent corresponding sides:

Equivalence

Theorem 9.15: *Two triangles are equivalent iff they have the same defect.*



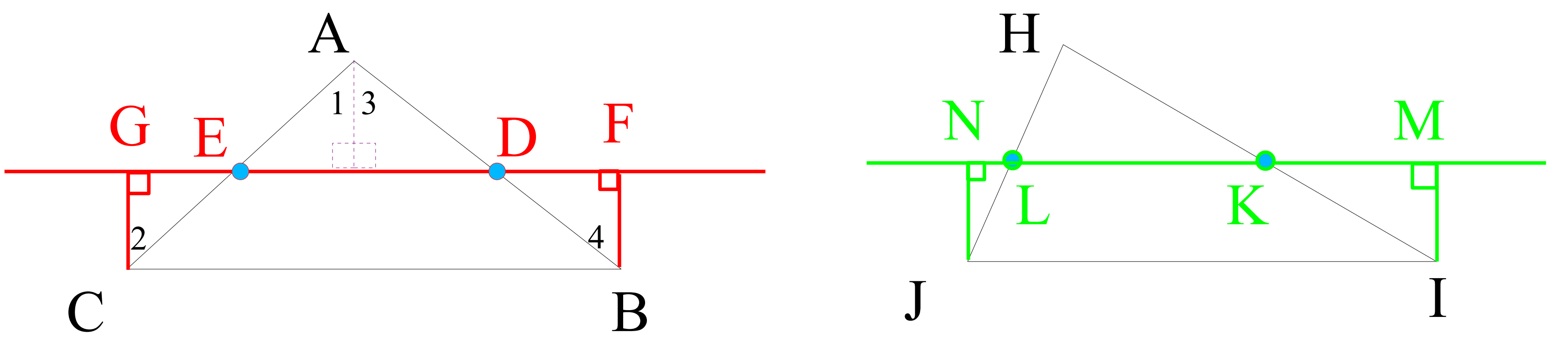
Suppose that triangles ABC and HIJ have the same defect and CB = JI.

In triangle ABC, let E be the midpoint of AC and D the midpoint of AB. By dropping perpendiculars from C and B to the line ED, meeting the line at points G and F respectively, we see that CGFB is a Saccheri quadrilateral and ABC is equivalent to it.

To see why the sides are congruent, drop altitude from A to ED. Both sides of the Saccheri quadrilateral are congruent to this altitude.

Equivalence

Theorem 9.15: *Two triangles are equivalent iff they have the same defect.*



Now do the same for triangle HIJ. It will be equivalent to Saccheri quadrilateral JNMI. Notice that the two Saccheri quadrilaterals have congruent summits (CB and JI respectively).

Note that the sum of the summit angles of CGFB is the angle sum of triangle ABC. The same is true of JNMI. Since the two triangles have the same defect, they must have the same angle sums. Therefore the two Saccheri quadrangles have congruent summit angles.

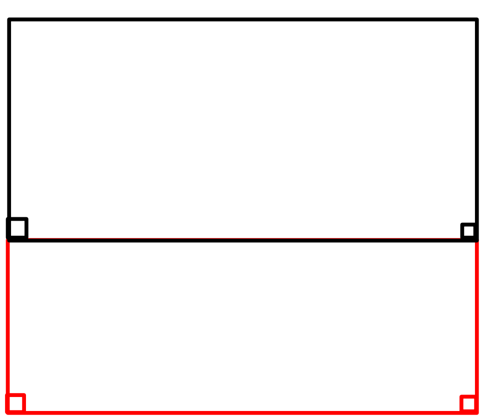
Equivalence

Theorem 9.15: *Two triangles are equivalent iff they have the same defect*.

We now need a little lemma:

Two Saccheri quadrilaterals are congruent if they have congruent summits and congruent summit angles.

Pf: On the same summit, construct two Saccheri quadrilaterals, having the same summit angles but different bases.



The figure on the bottom is a rectangle and these do not exist in hyperbolic geometry. So the two quadrangles have the same base and are thus congruent.

Equivalence

Theorem 9.15: *Two triangles are equivalent iff they have the same defect*.

Thus, our two Saccheri quadrilaterals are congruent. Since each triangle is equivalent to its own Saccheri quadrilateral, they must be equivalent to each other.

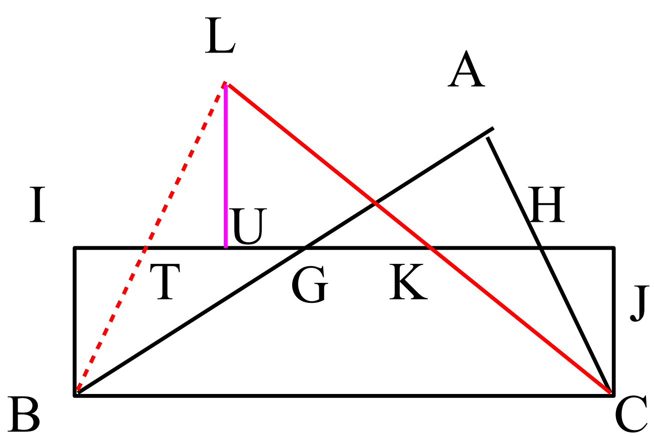
Now we turn to the case where the two triangles have the same defect, but no corresponding congruent sides.

Let ABC and DEF be two such triangles with the same defect, and suppose wlog that DE > AC.

With G and H the midpoints of AB and AC respectively, we construct the Saccheri quadrilateral which is equivalent to ABC as we have done before.

Equivalence

Theorem 9.15: *Two triangles are equivalent iff they have the same defect.*



Draw LC congruent to DF (not drawn) so that its midpoint K lies on IJ. Connect L and B, drop perpendicular from L to IJ.

Triangle LUK ≅ triangle CJK (AAS). Thus, LU = JC (= IB). Now, triangle IBT ≅ triangle ULT (AAS), and so, LT = TB. Since T and K are the midpoints of LB and LC, quadrangle BIJC is equivalent to triangle BLC. Since, ABC and BLC are equivalent to the same quadrangle, they are equivalent to each other, so BLC has the same defect as ABC and so, DEF. But BLC also has a side congruent to one of DEF, so they are equivalent and both equivalent to ABC.

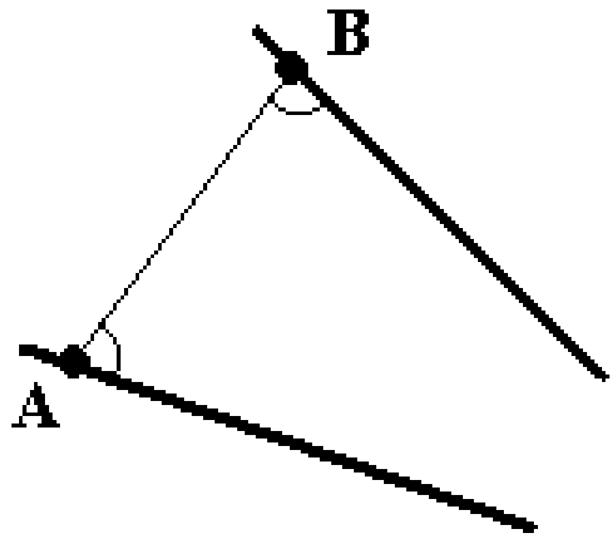
Area of a Triangle

Def: The ***area of a triangle*** is $A = kd$, where d is the defect and k is a positive constant that is the same for all triangles. (Equivalent polygons will then have the same area).

The constant k depends on a choice of a particular triangle chosen to have unit area. Note that the larger (in area) the triangle, the greater the defect (and hence the smaller the angle sum). Because there is a maximum defect ($= \pi$) there will be a maximum area for any triangle in the geometry.

Curves

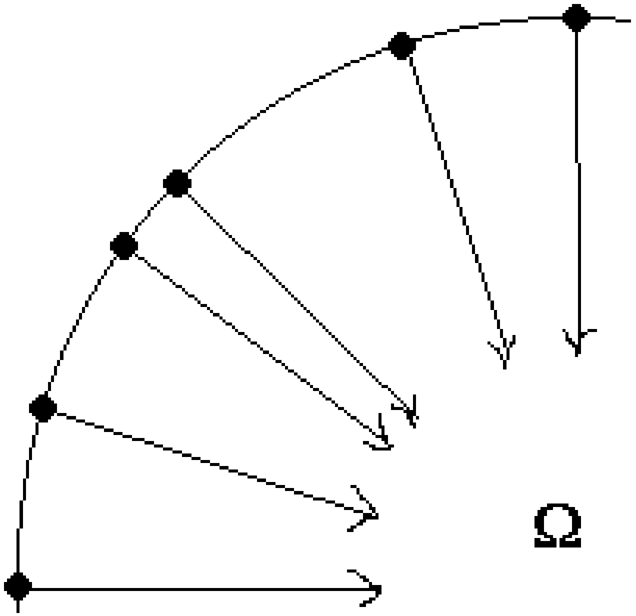
Def: Two points, one on each of two lines, are called ***corresponding points*** if the two lines form congruent angles on the same side with the segment whose endpoints are the two given points. (This definition is valid if the two lines meet at an ordinary point, an omega point or a gamma point).



Corresponding Points

Curves

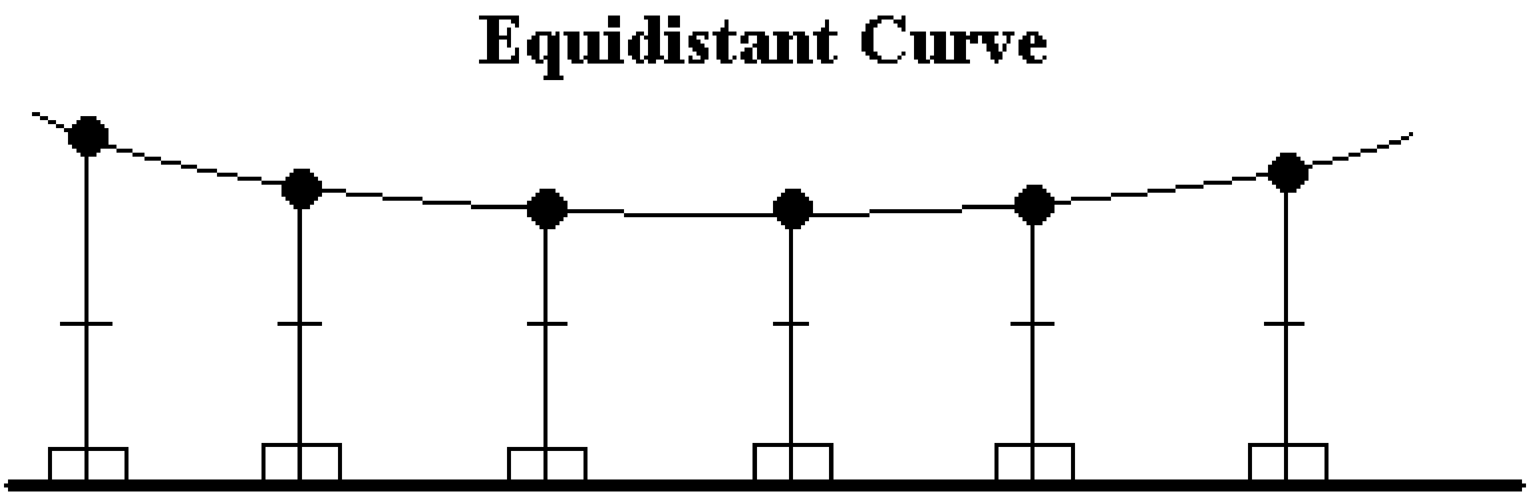
Def: A ***limiting curve*** (***horocycle***) is the set of all points corresponding to a given point on a pencil of rays with an ideal point as vertex.



Limiting Curve

Curves

Def: An ***equidistant curve*** is the set of all points corresponding to a given point on a pencil of rays with a common perpendicular.



Theorems

Limiting curves behave very much like circles (analogous to the way that omega triangles behave like triangles). For instance, it can be proved that:

Theorem 9.16: *Three distinct points on a limiting curve uniquely determine it.*

Note that this does not say that any three points (non-collinear) determine a limiting curve ... the points must already be on a limiting curve.

Theorems

However, there are some properties of limiting curves which are quite different from those of circles in Euclidean geometry.

Theorem 9.17: *Any two different limiting curves are congruent*.

Elliptic Geometry

Characteristic Postulate of Elliptic Geometry

Any two lines in a plane meet at an ordinary point.

Other modifications of Euclidean axioms are needed to get a consistent set of axioms for this geometry. These include:

1. Lines are boundless rather than infinite.
2. Circles do not have to always exist.

Theorems

Theorem 9.18: The segment joining the midpoint of the base and summit of a Saccheri quadrilateral is perpendicular to both the base and summit.

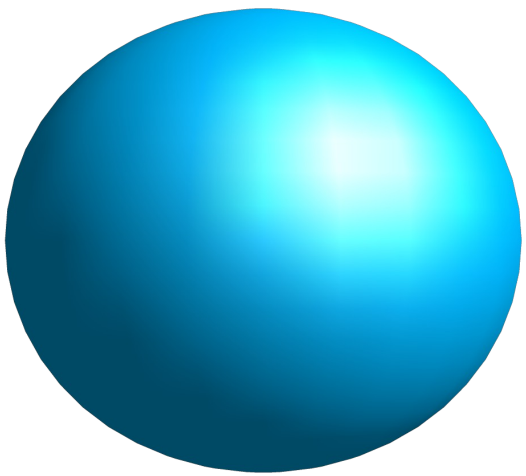
Theorem 9.19: The summit angles of a Saccheri quadrilateral are congruent and obtuse.

Theorem 9.20: A Lambert quadrilateral has its fourth angle obtuse, and each side of this angle is shorter than the opposite side.

Theorem 9.21: The sum of the measures of the angles of any triangle is greater than π.

Theorem 9.22: The sum of the measures of the angles of any quadrilateral is greater than 2π.

A Model for Elliptic Geometry



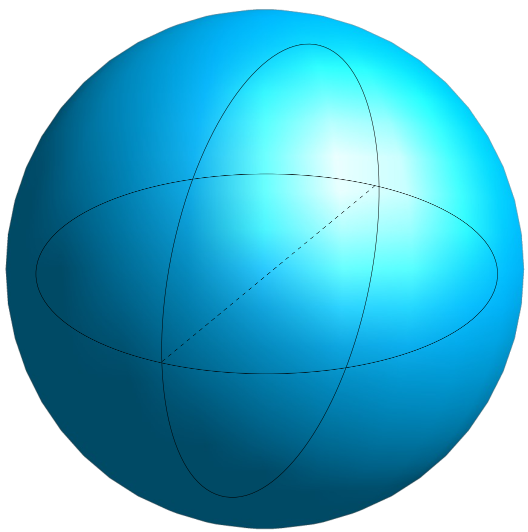
POINTS: Pairs of antipodal points on the surface of a sphere.

We identify a point with the point that is diametrically opposite from it (i.e., North pole is identified with South pole, etc.)

LINES: Great circles of the sphere.

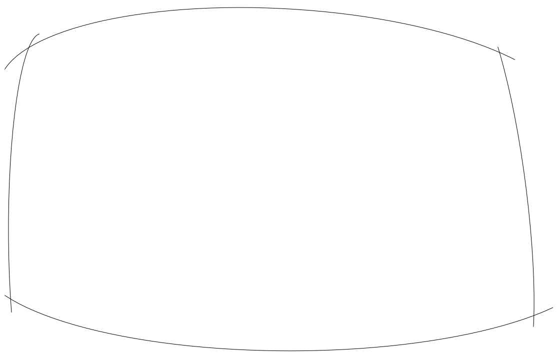
A great circle (like the equator of the sphere) is the intersection of the sphere and a plane which passes through the center of the sphere.

A Model for Elliptic Geometry



The characteristic postulate for elliptic geometry.

A Saccheri Quadrilateral



Consistency

Recall that an axiom system is said to be consistent if no logical contradictions can be derived from these axioms. Proving that a system is consistent is frequently too difficult to be done. We are often satisfied if we can show that one axiom system is "just as" consistent as another, well known system, even if we can't prove that either are absolutely consistent.

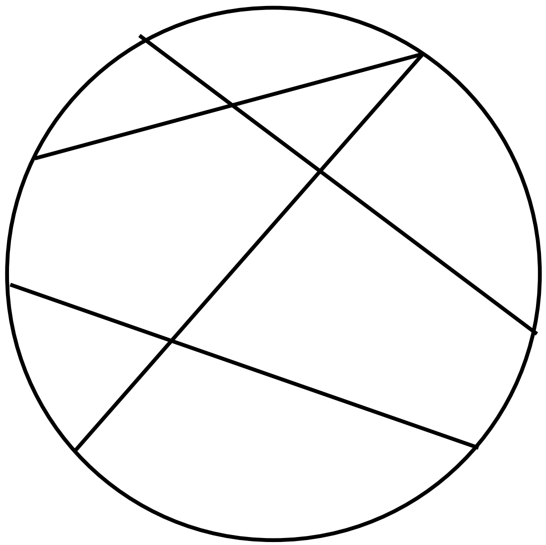
Relative Consistency

Def: Two axiom systems are said to be ***relatively consistent*** if any contradiction derived in one of the systems implies that there is a contradiction that can be derived in the other.

The relative consistency of axiom systems can be proved by providing a model of one system inside the other. In terms of the geometries we have studied, if we can model, say, hyperbolic geometry, using only the objects and relations of Euclidean geometry, then hyperbolic geometry and Euclidean geometry would be relatively consistent since the model would permit translation of any contradiction in one system into a contradiction in the other system. E. Beltrami (1868) is credited with doing this for the first time. Since then several other models have been created. Thus, hyperbolic (and elliptic as well) geometry is relatively consistent with Euclidean geometry. Another way to say this is that all three geometries are equally logically valid.

Models for Hyperbolic Geometry

Klein (1849-1925) model for hyperbolic geometry.



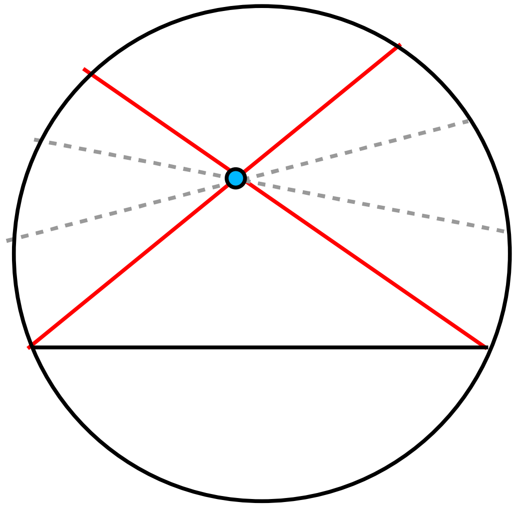
POINTS: The interior points of a fixed circle.

LINES: The chords of this circle.

Omega (ideal) points are the points on the circle (which are not in the geometry). Thus, parallel lines are those which meet on the circle.

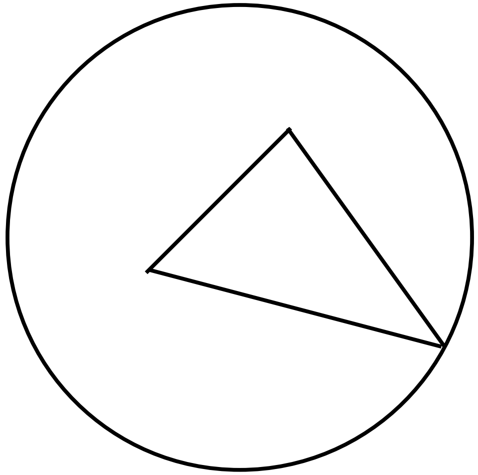
Gamma (ultra-ideal) points are the points outside of the circle.

The Klein Model



The characteristic postulate:

An omega triangle:



The Klein Model

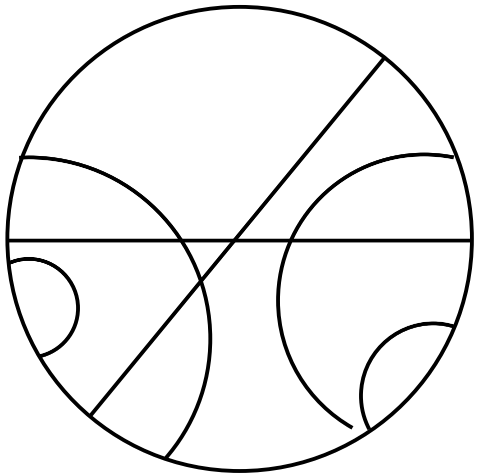
Problems with the Klein model

a) Distances are not represented correctly.
- chords should have infinite lengths.

b) Angles are not represented correctly.
- one can easily construct "Euclidean" rectangles in the model .... but these don't exist.

Models for Hyperbolic Geometry

Poincaré (1854-1912) model for hyperbolic geometry.

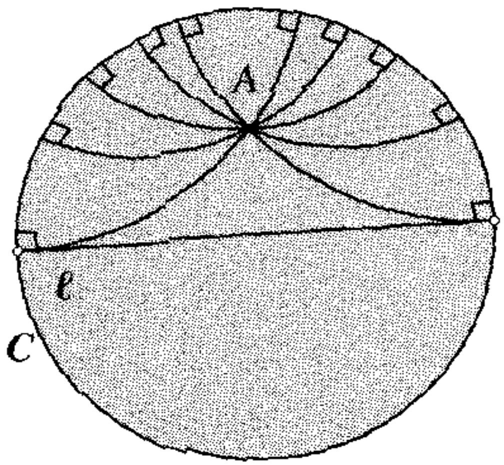


POINTS: The points interior to a fixed circle.

LINES: Circular arcs of circles which are orthogonal to the fixed circle and straight lines (chords) through the center of the fixed circle.

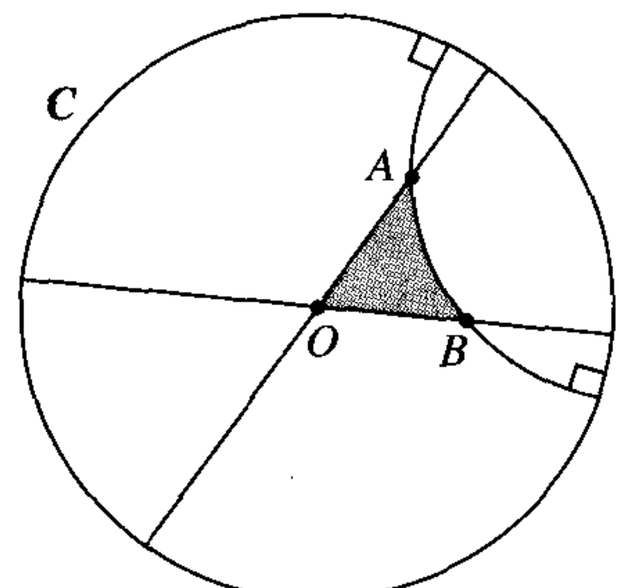
Ideal points are the points on the fixed circle.
Gamma points are not well represented in this model.

Poincaré Model

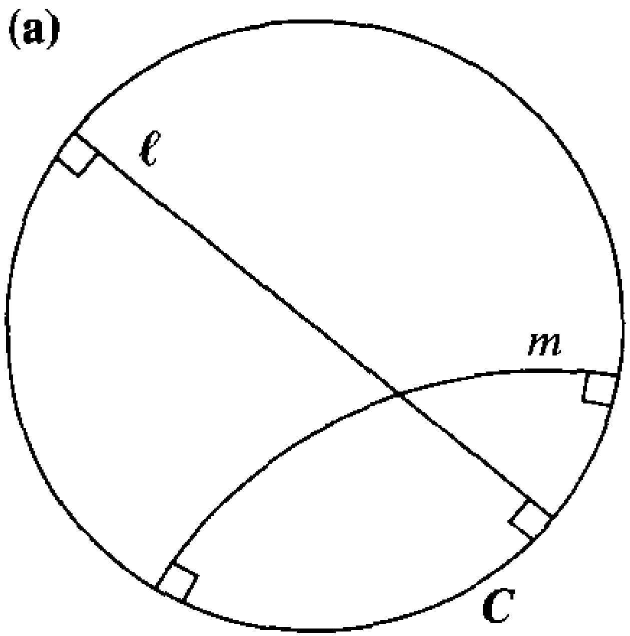


The characteristic postulate.

An ordinary triangle with center as a vertex.

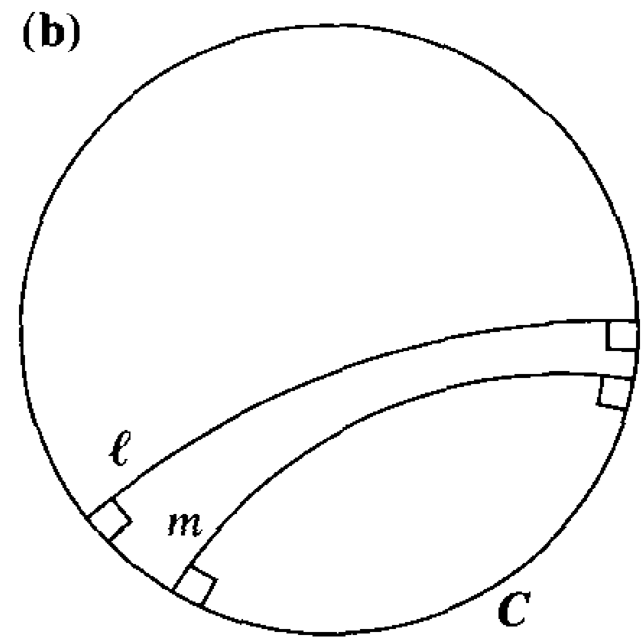


Poincaré Model

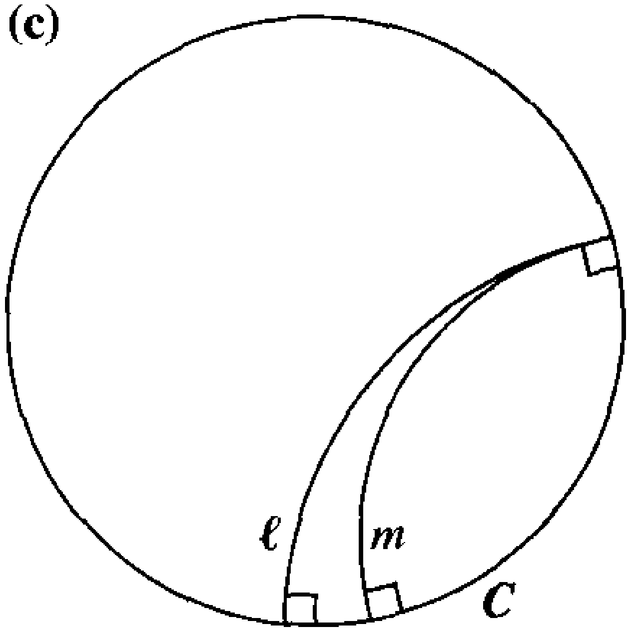


Intersecting lines.

Non-intersecting Lines

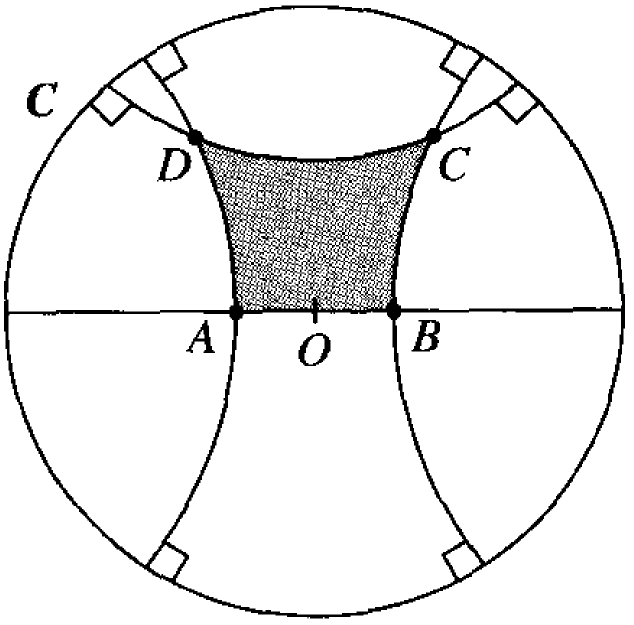


Poincaré Model



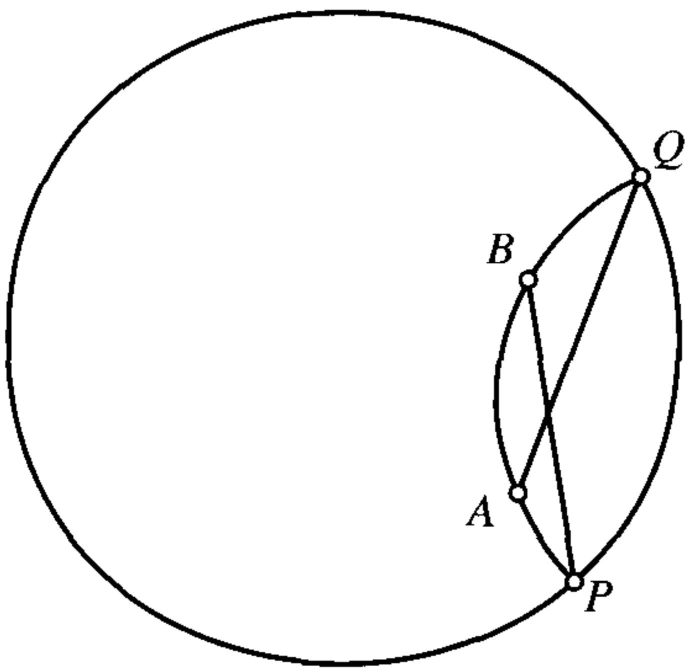
Parallel lines.

A Saccheri quadrilateral.



Distance in the Model

$$d(A,\ B) = \left|\ln\left(\frac{AQ}{BQ} \times \frac{BP}{AP}\right)\right|$$

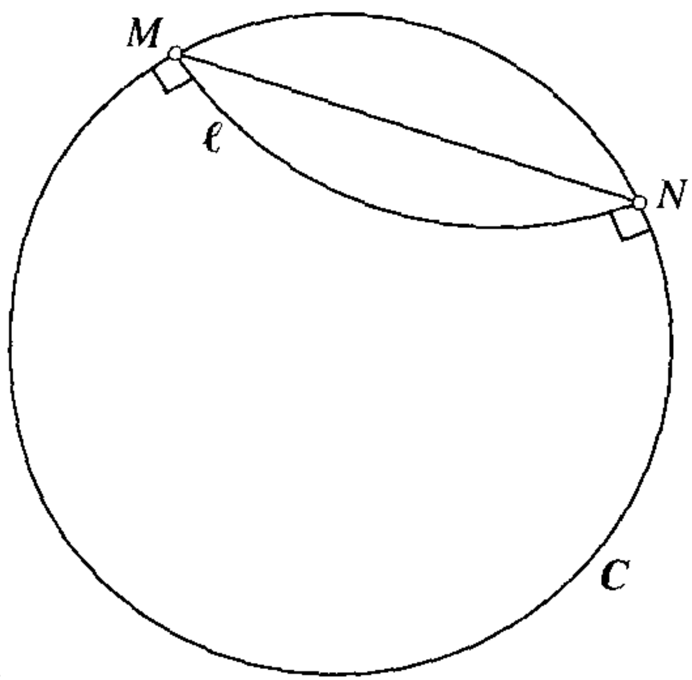


As A and B approach P and Q respectively, the length of segment AB gets larger and larger without bound.

Think of a "ruler" which shrinks as it moves closer to the rim of the model.

The idea is a modification due to F. Klein of one due to A. Cayley.

Poincaré Model



An equidistant curve to the line ℓ is represented in the model by the straight line segment MN.