Geometric Constructions

Philosophy of Constructions

Constructions using compass and straightedge have a long history in Euclidean geometry. Their use reflects the basic axioms of this system. However, the stipulation that these be the only tools used in a construction is artificial and only has meaning if one views the process of construction as an application of logic. In other words, this is not a **practical** subject, if one is interested in constructing a geometrical object there is no reason to limit oneself as to which tools to use.

Philosophy of Constructions

The value of studying these constructions lies in the rich supply of problems that can be posed in this way. It is important that one be able to analyze a construction to see why it works. It is *not* important to gain the manual dexterity needed to carry out a careful construction.

Compass vs. Dividers

The ancient Greek tool used to construct circles is *not* the modern day compass. Rather, they used a device known as a divider. Dividers consist of just two arms with a central pivot. Should you pick up a divider, the arms will collapse, so it is impossible to use them to transfer lengths from one area to another. Modern compasses remain open when picked up, so such transfers are possible. Given the difference in the two tools, it appears that the modern compass is a more powerful instrument, capable of doing more things.

Compass vs. Dividers

However, this is not true.

The ancient dividers can do everything that modern compasses can. Of course, this means that how certain constructions were done by the ancient Greeks are quite different from the way we would do them today. This underscores the statement above; technique is not as important as understanding why it works.

Basic Constructions

The basic constructions are:

1. Transfer a segment.
2. Bisect a line segment.
3. Construct a perpendicular to a line at a point on the line.
4. Construct a perpendicular to a line from a point not on the line.
5. Construct an angle bisector.
6. Copy an angle.
7. Construct a parallel to a line through a given point.
8. Partition a segment into n congruent segments.
9. Divide a segment into a given ratio (internal and external).

Basic Construction 1

Transfer a line segment.

A'

A   B
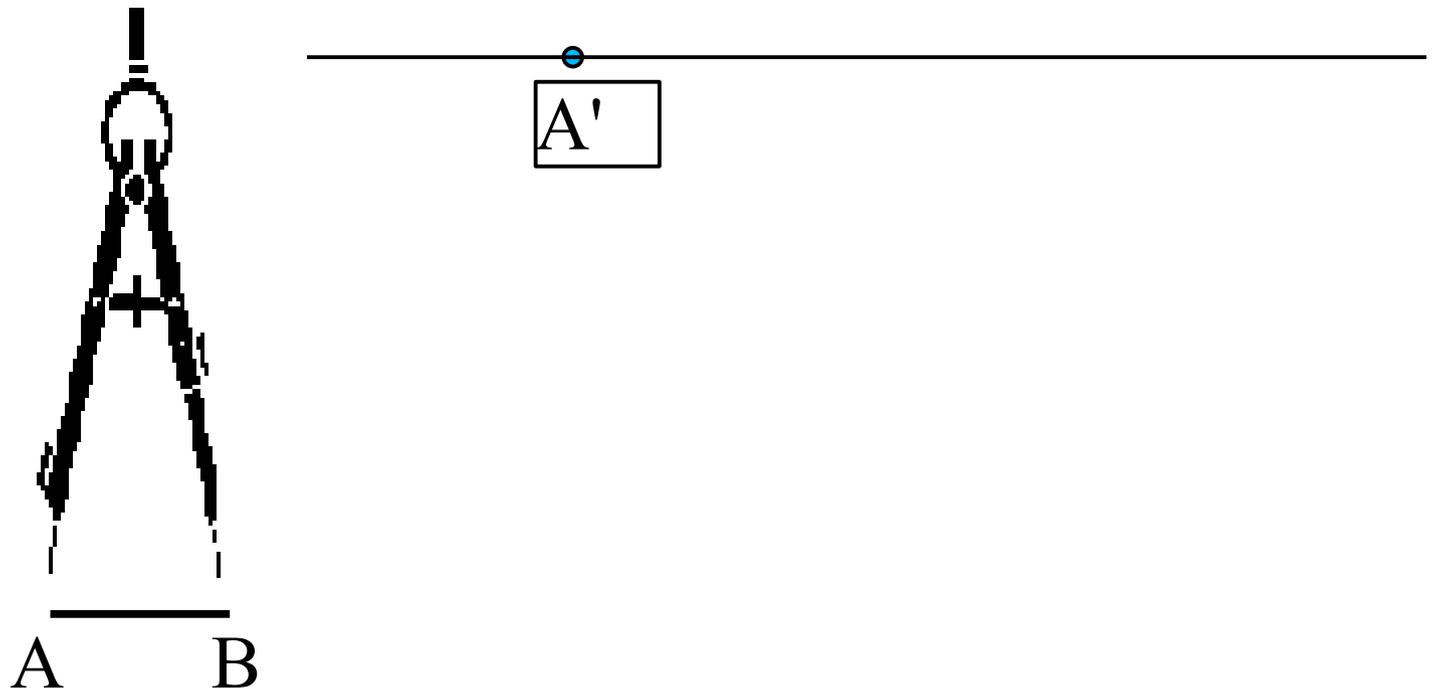
Basic Construction 2

Bisect a line segment

$\overline{}$
A   B

Step 1: With compass set to more than half the length, draw an arc with center A.
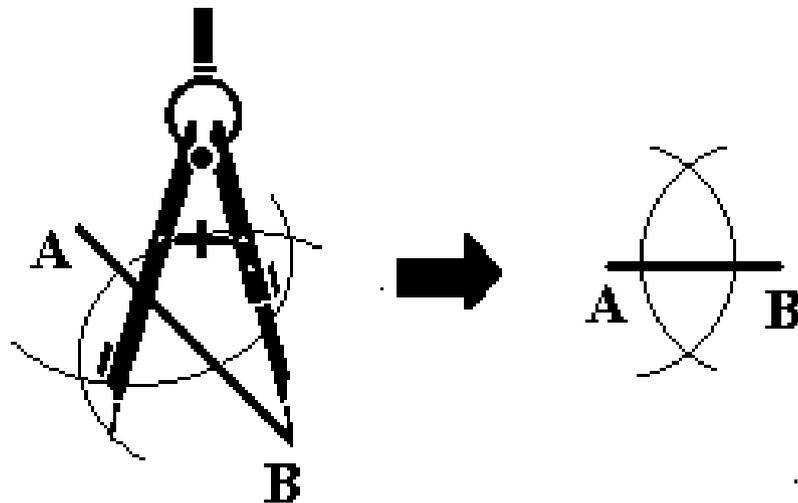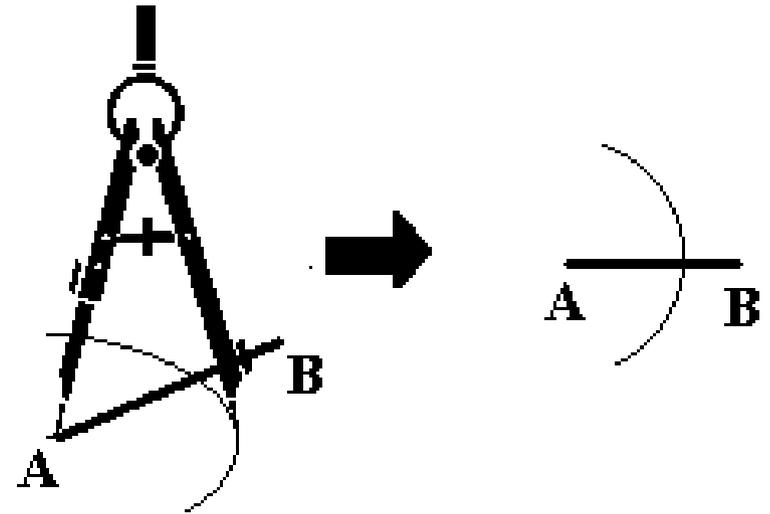
Step 2: With the same compass setting draw another arc with center B. Be sure that the two arcs meet at two points.

Step 3: Join the intersection points of the arcs with a straightedge. This line bisects the original segment.
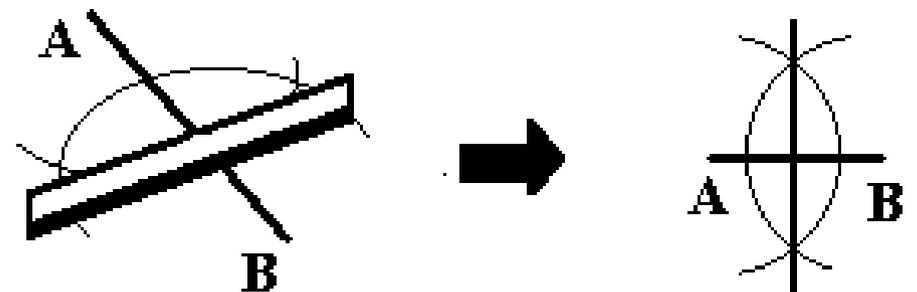
Basic Construction 3

Construct a perpendicular to a line from a point on that line.

Step 1: Set compass to any radius. Draw an arc with center P that intersects the line twice, at points A and B.
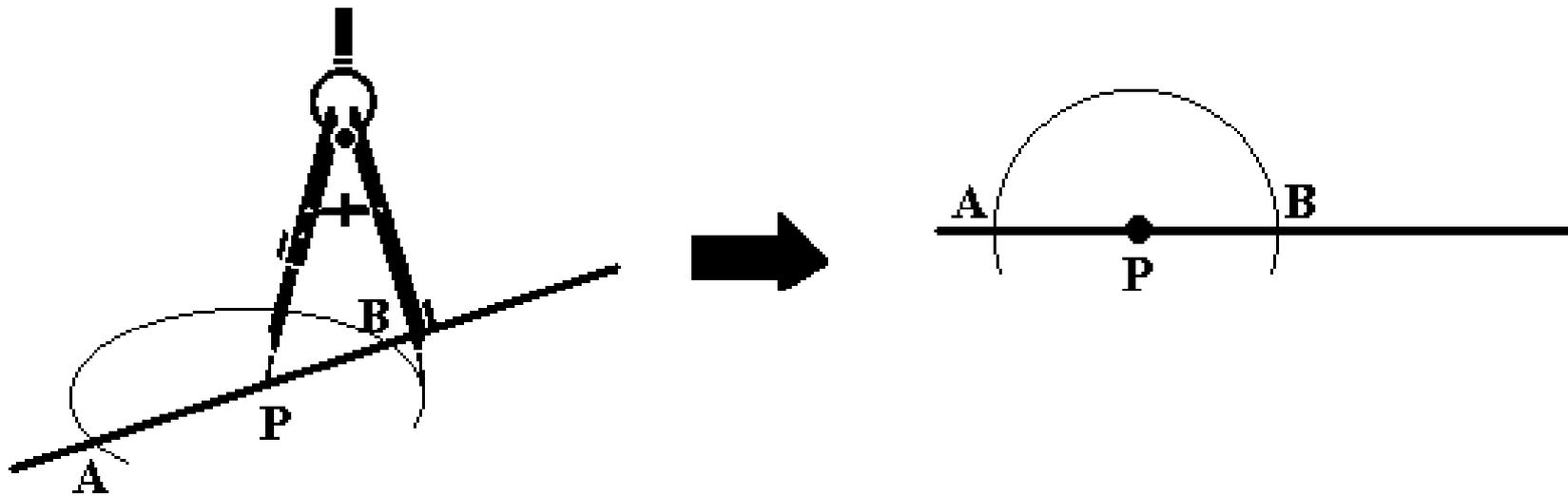
Step 2: Follow directions for bisecting the line segment AB.
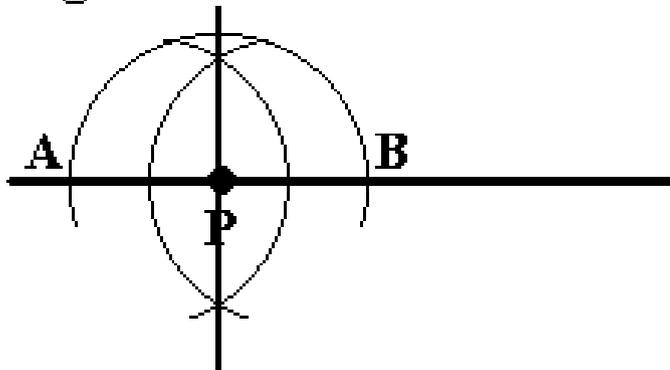
Basic Construction 4

Construct a perpendicular to a line from a point not on that line.

P ●

_____

Step 1: Set compass to any radius. Draw an arc with center P that intersects the line twice, at points A and B.
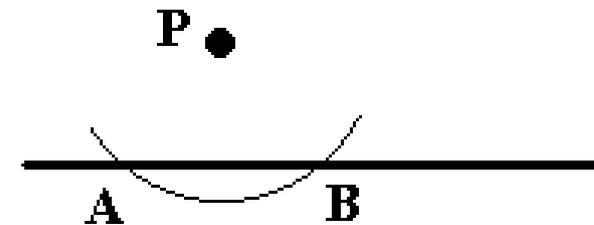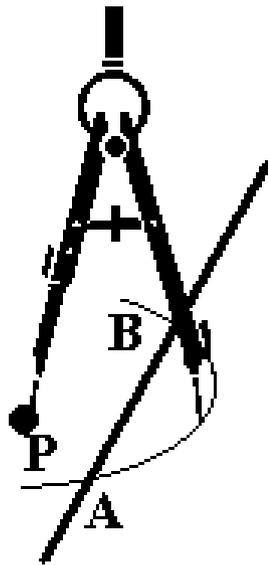
Step 2: Follow directions for bisecting the line segment AB.
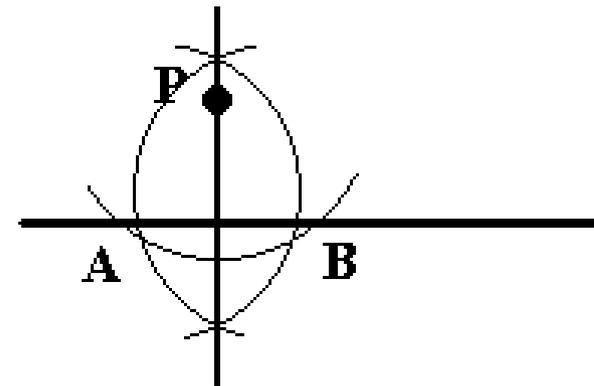
Basic Construction 5

Construct the angle bisector of a given angle.
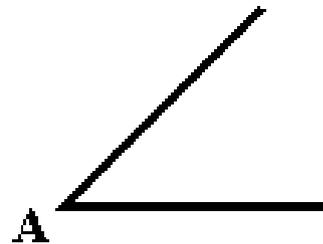
Step 1: Draw an arc with center A of any radius.
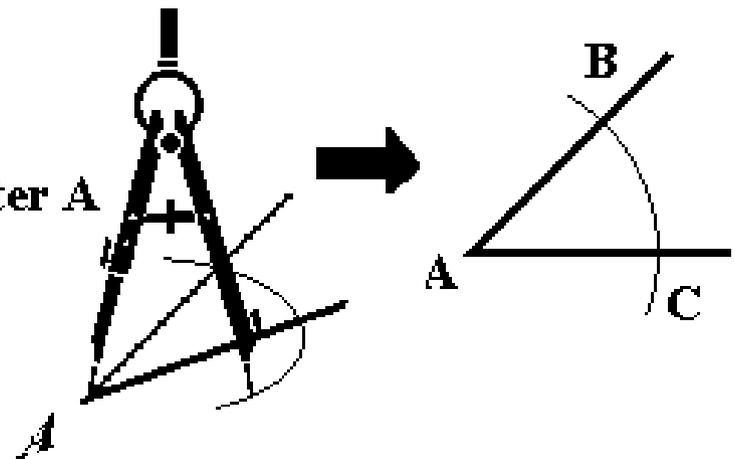
Step 2: Draw an arc with center B of any radius greater than half of BC. Repeat this with center C using the same radius. Make sure the two arcs cross.
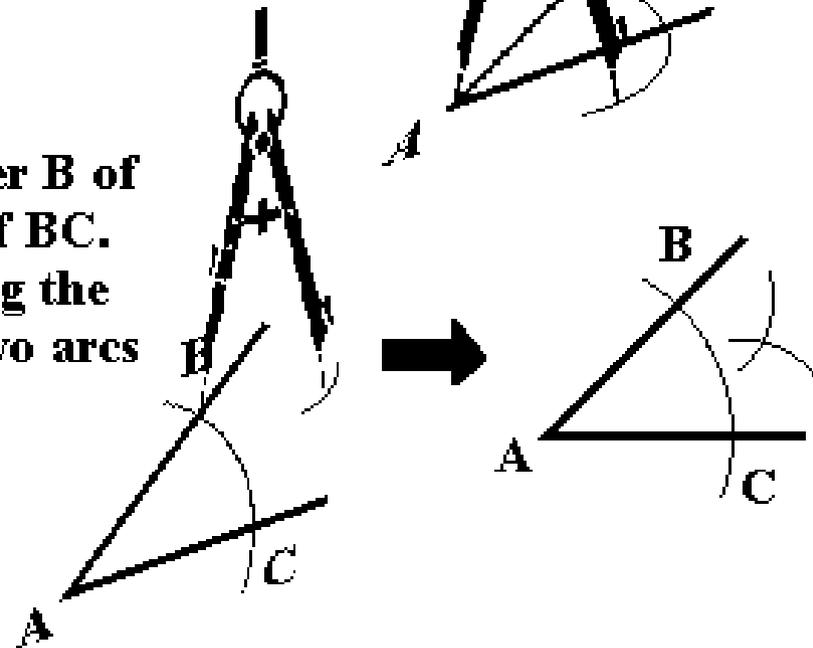
Step 3: Join A to the point where the arcs cross.
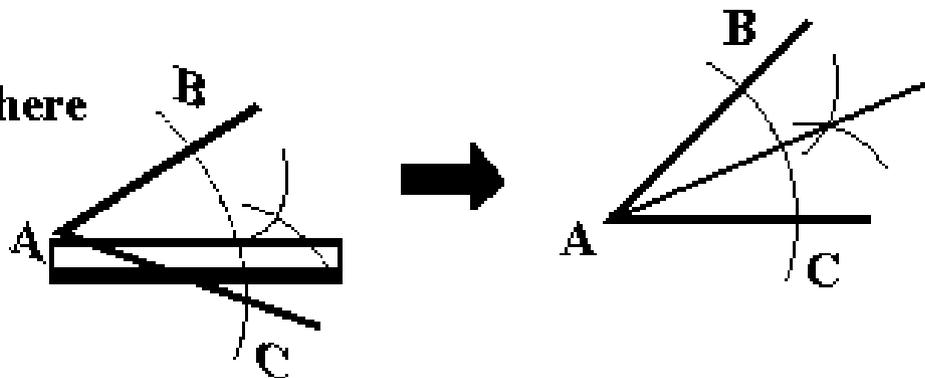
Basic Construction 6

Copy an angle

Step 1: Draw an arbitrary arc with center A of the given angle, then with the same compass setting redraw the arc with center A'.
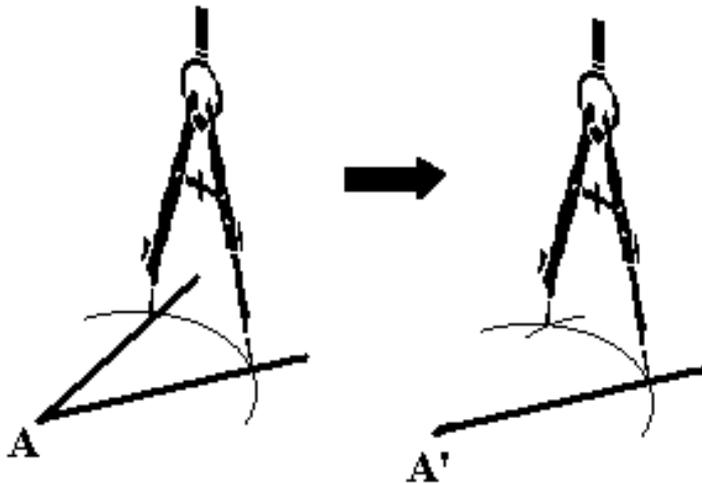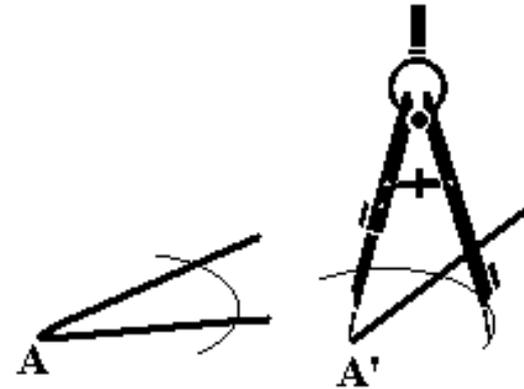
Step 2: Measure the arc between the sides of the original angle, then mark off this measure on the arc with center A'.

Step 3: Join the mark to A'.
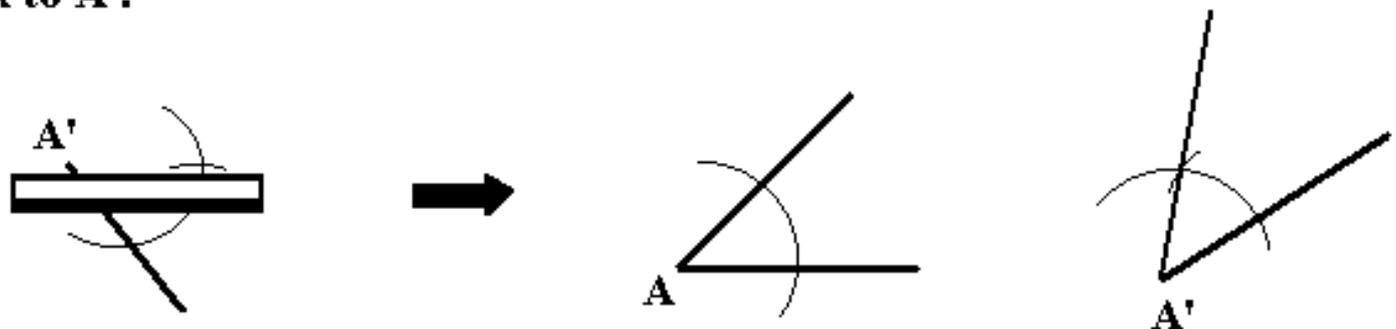
Basic Construction 7

Construct a parallel to a given line through a point not on that line.

P .
_____

Step 1: Draw any line through P that meets the given line.
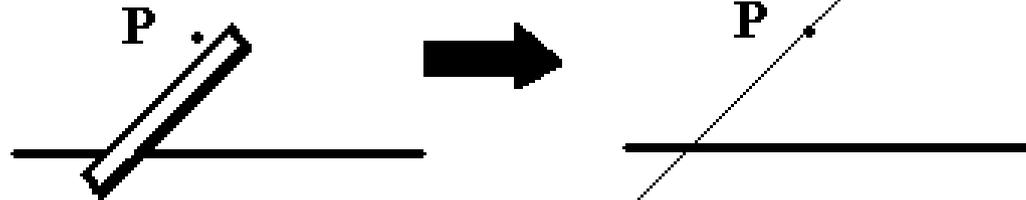
Step 2: Copy the angle at A on the other side of the line just drawn with vertex at P.
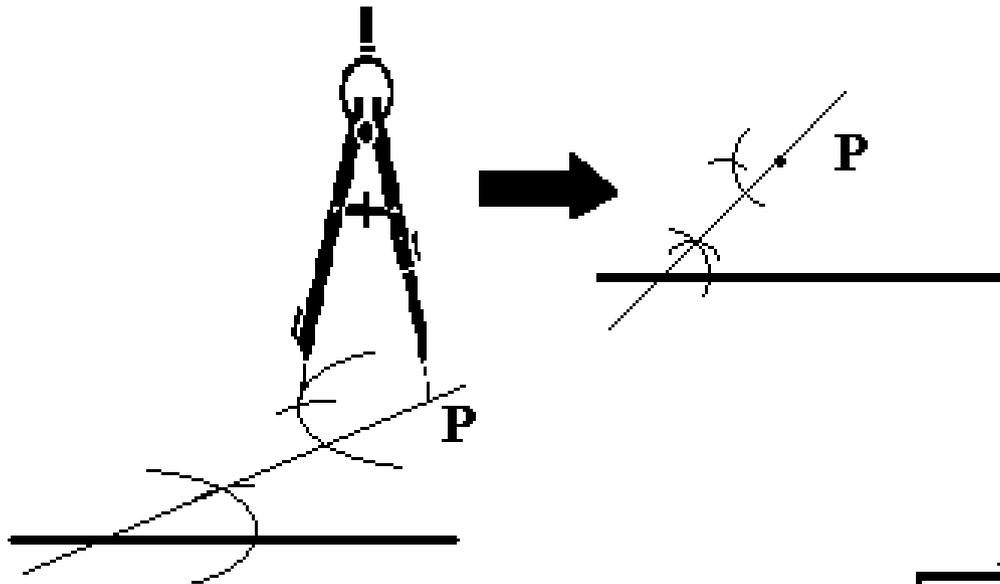
Step 3: Extend the side of the new angle through P, giving the desired parallel.
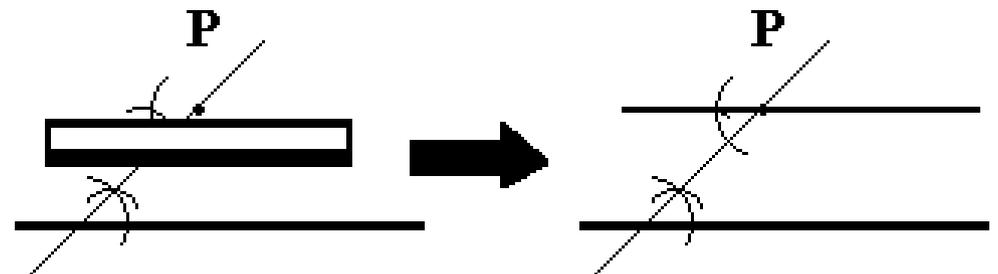
Basic Construction 8

Divide a line segment into n equal parts

A ———————————— B

Step 1: Draw any line through A.
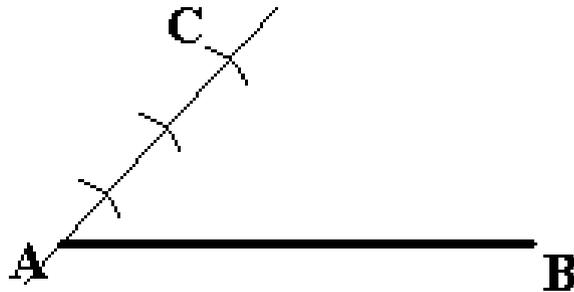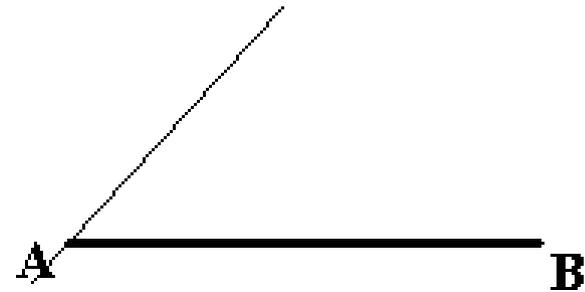
Step 2: Using any compass setting, mark off n equally spaced segments along this line. Ending at point C.

Step 3: Draw the line segment BC.
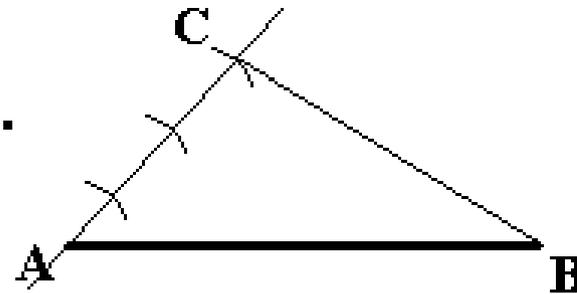
Step 4: Through each of the marks on AC construct a parallel to BC and extend to intersect AB.
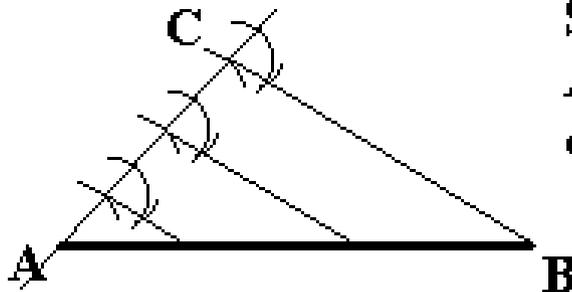
Basic Construction 9

To partition segment AC internally and externally in the ratio of 3 to 2, given a unit segment:

1. Draw any line other than AC through A, construct a parallel to this line through C.

2. Mark off 3 unit segments on the line through A, and 2 unit segments on the line through C (in both directions from C).
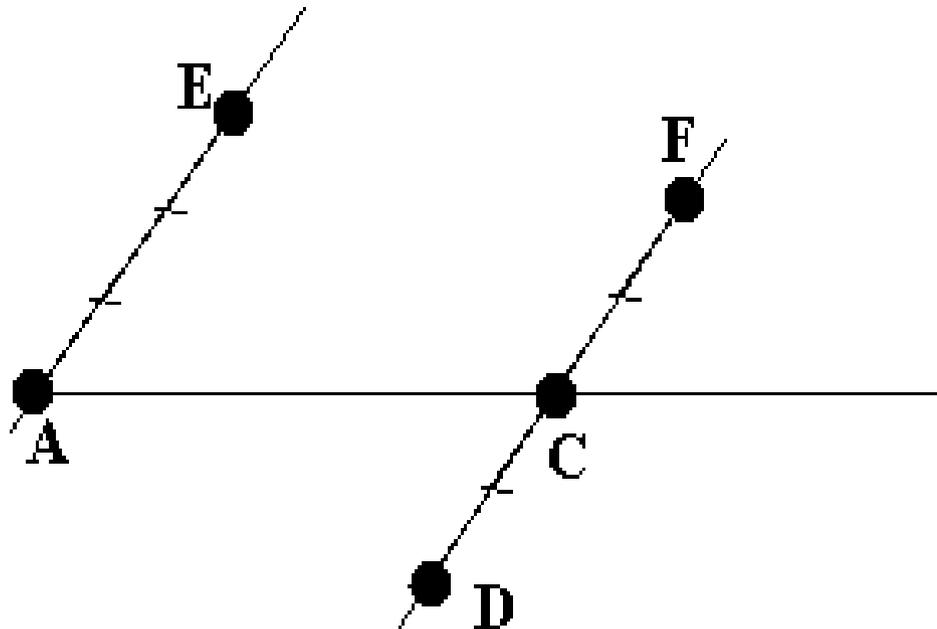
Basic Construction 9

3. Join E and D to get B, the internal divider and join E and F to get G, the external divider.
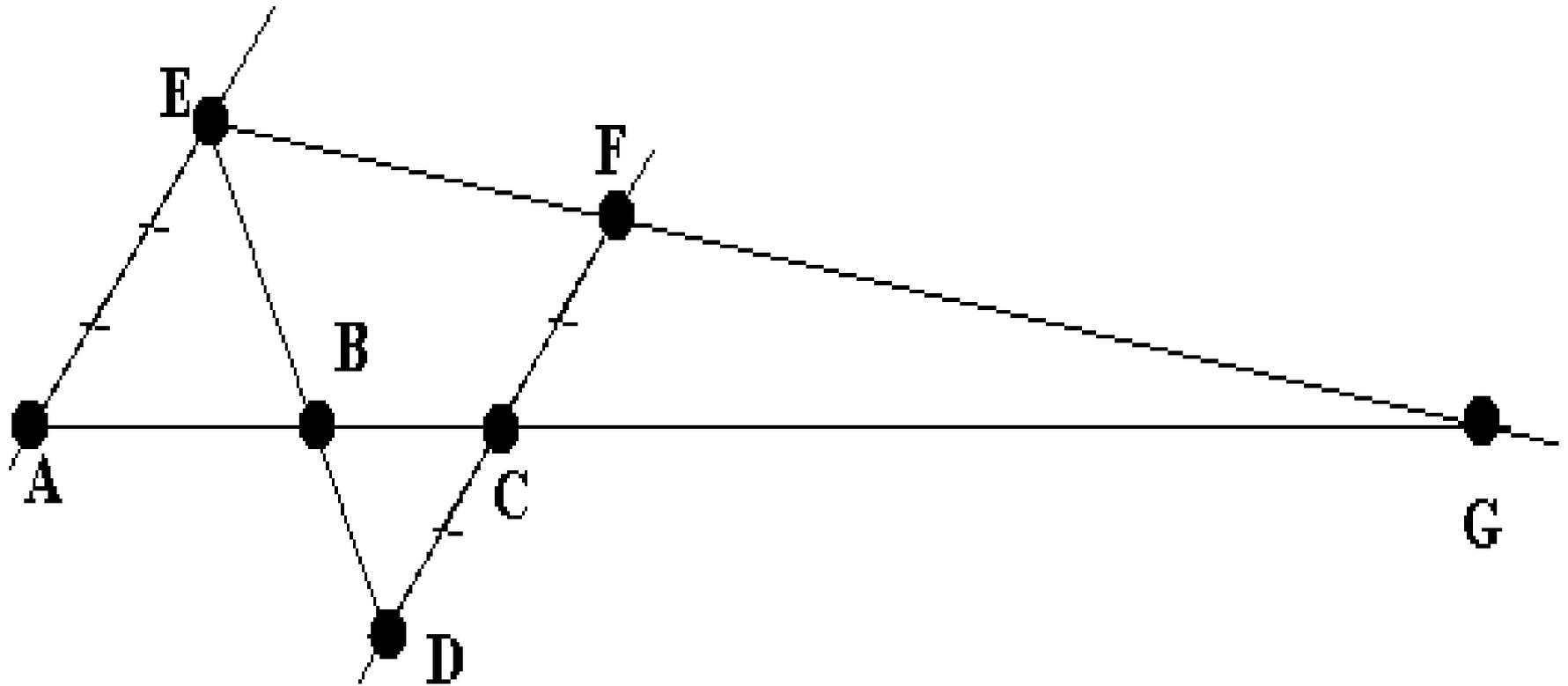
Constructible Numbers

Given a segment which represents the number 1 (a ***unit segment***), the segments which can be constructed from this one by use of compass and straightedge represent numbers called ***Constructible Numbers***. Note that the restrictions imply that the constructible numbers are limited to lying in certain quadratic extensions of the rationals.

Given two constructible numbers one can with straightedge and compass construct their:

Sum

Difference

Product

Quotient

Square Root

Constructible Numbers

Sum

a
_____

b
_____

$$\text{\color{red}a} \quad + \quad \text{\color{green}b}$$

Difference

a
_____

b
_____

$$\text{\color{green}b} - \text{\color{red}a}$$
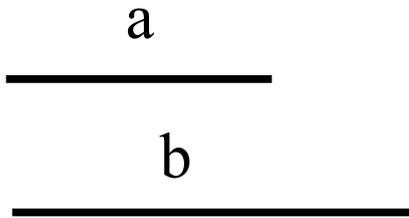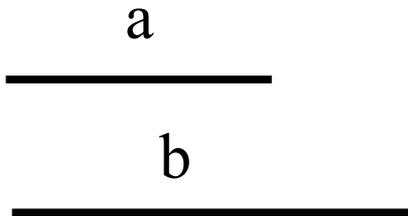
Constructible Numbers

Product

1

a

b

ab

b

1
a

Quotient

1

a

b

a

a/b

1
b
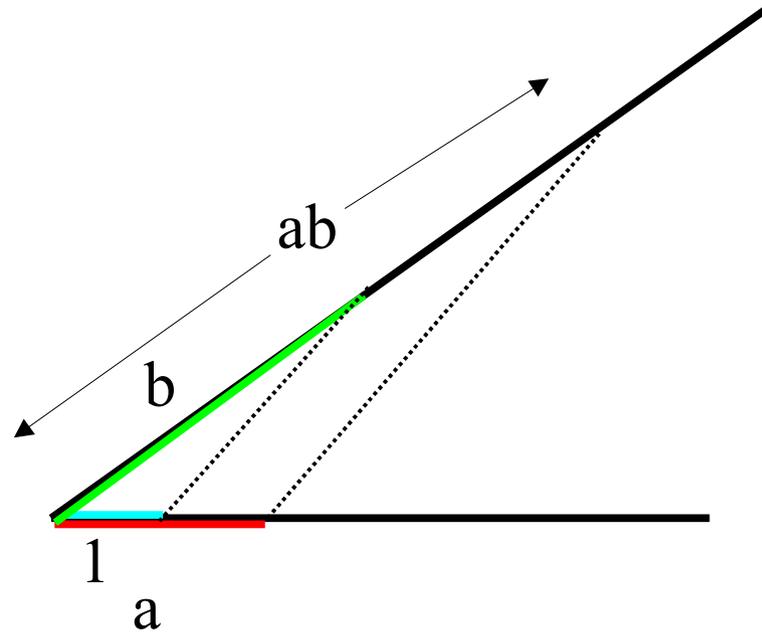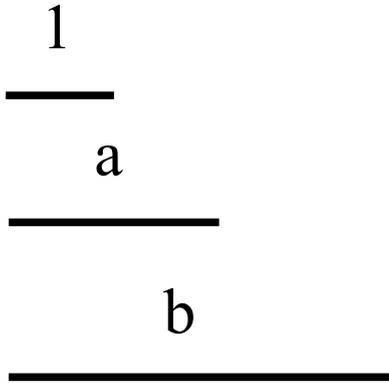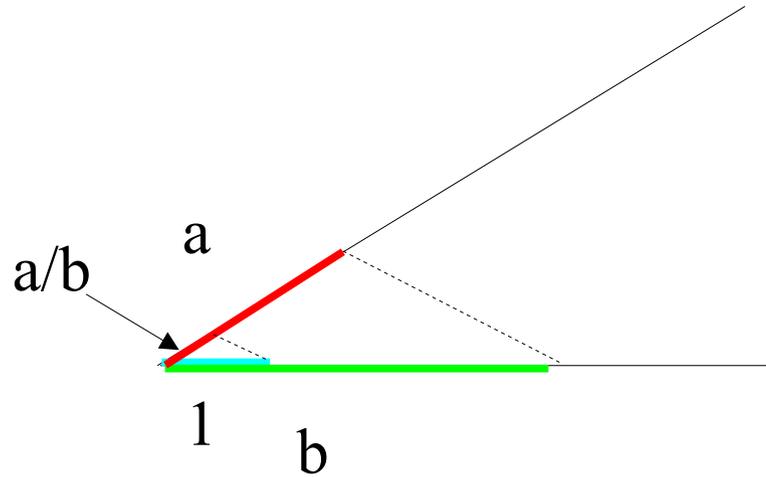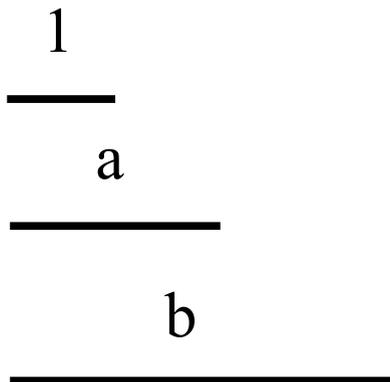
Constructible Numbers

Square Root

$$\frac{1}{a}$$

$$\sqrt{a}$$

a   1
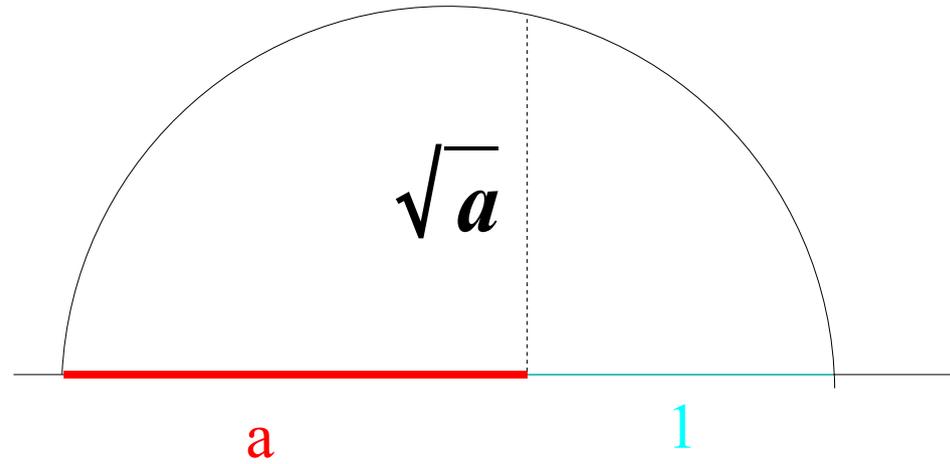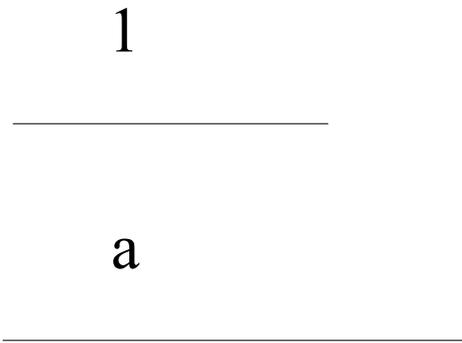
Constructions

Example: Construct a triangle, given the length of one side of the triangle, and the lengths of the altitude and median to that side.

a
length of side

b
length of median

c
length of altitude

Step 1: Take any line and mark off the length a. Construct the perpendicular bisector of this line segment.
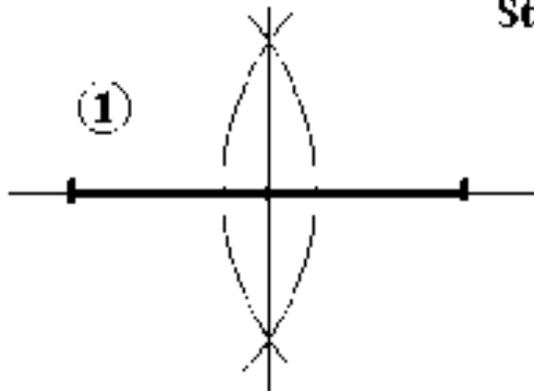
Step 2: Using the midpoint of the segment as center and radius b, draw a circle.

① ② ③
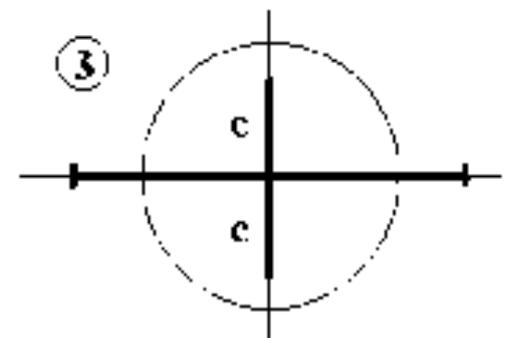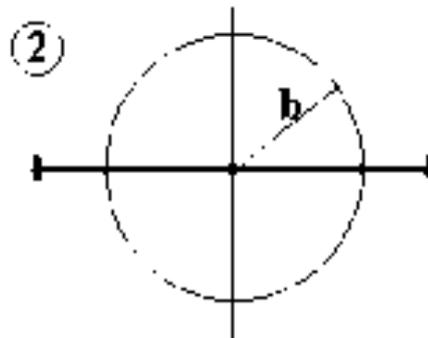
Constructions

Step 3: Starting at the midpoint, measure length c along the perpendicular bisector (going either up or down).

Step 4: Construct the parallel to the original line segment at the point at distance c from that line segment.
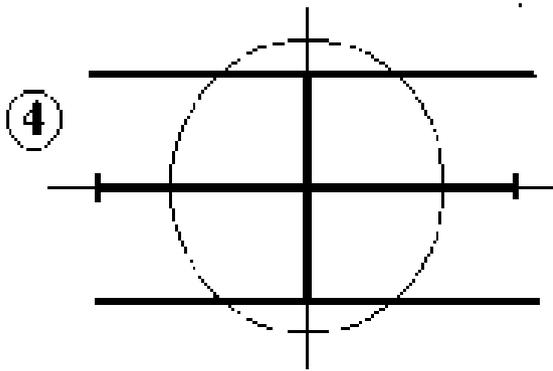
Step 5: Any point on the circle and one of the parallels can be used as the third vertex of the triangle.
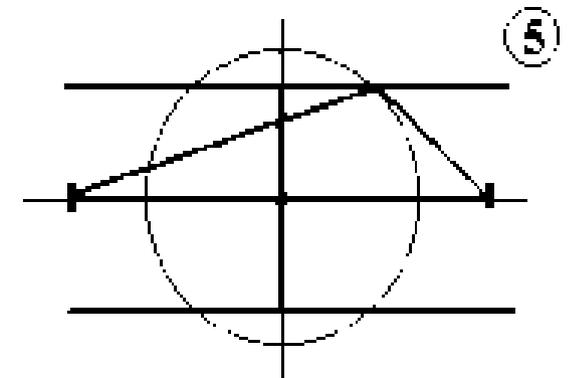
As the third vertex is determined by the intersection of one of two parallel lines with a circle, there are three possibilities for the number of solutions. If b is less than c, there will be no intersection, so no solutions. If b equals c, the lines will be tangent to the circle and we would get two solutions. Finally, if b is greater than c (the situation drawn above) then there will be four points of intersection.

Constructions

Example: Construct a triangle, given one angle, the length of the side opposite this angle, and the length of the altitude to that side.

a

length of side

c

length of altitude
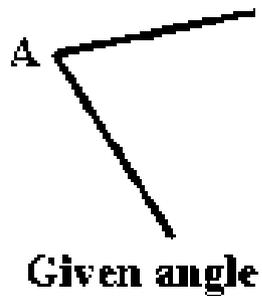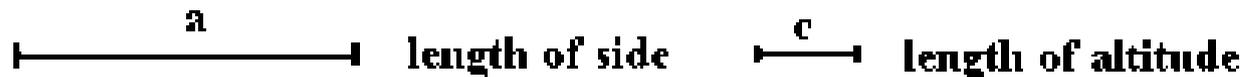
Step 1: Form any triangle with angle A and opposite side of length a by setting the compass to radius a and drawing the arc from any point on one side of the angle to the other side.

A

Given angle

A

①

a
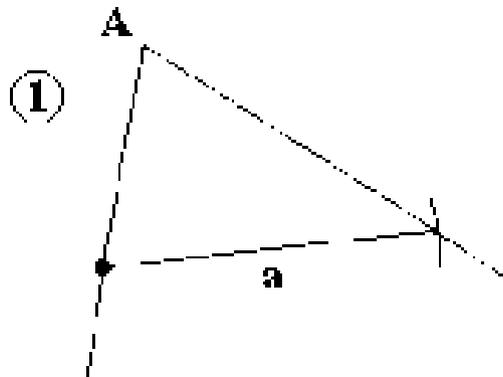
Step 2: Construct circumcircle of this triangle.

②
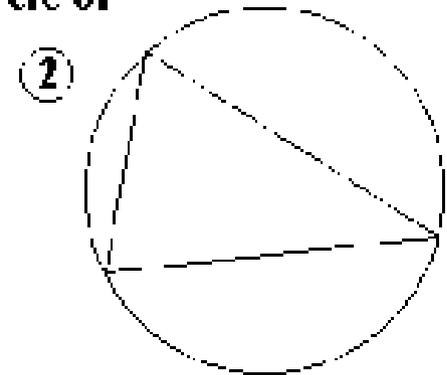
Step 3: At any point on side a, construct a perpendicular and measure length c on the same side of side a as the angle.

Constructions
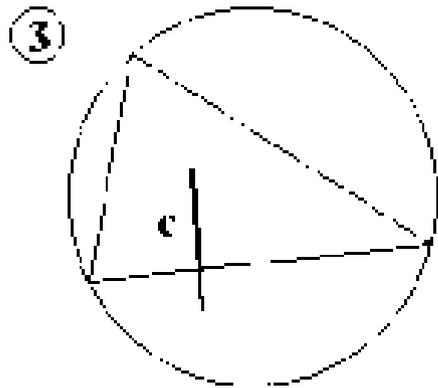
Step 4: Construct the line parallel to side a through the point at distance c from it.

Step 5: Any intersection of this line with the circle can be used as the location of A.
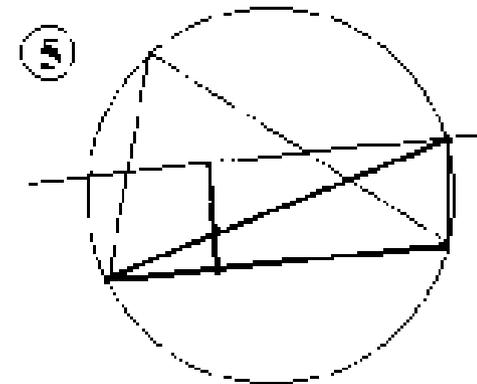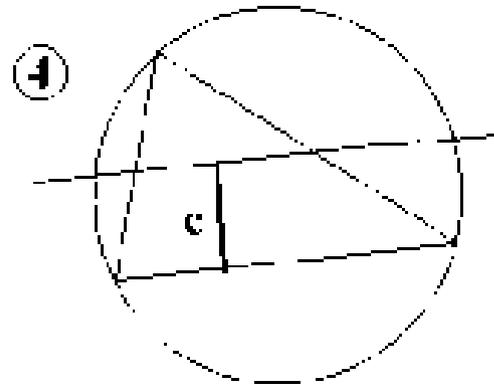
As the position of vertex A is determined by the intersection of a single line with a circle, there are three possibilities for the number of solutions. If the parallel does not intersect the circle, there is no solution. If the parallel is tangent to the circle there is one solution, and finally, if the parallel intersects the circle twice, there are two solutions (as indicated in the situation drawn above).

Constructions

Example: Construct a triangle, given the circumcenter O, the center of the nine-point circle N, and the midpoint of one side A'.

Given:
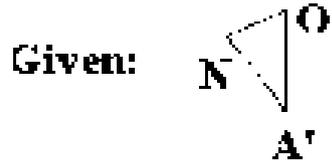
Step 1: Extend A'N to point J so that A'N – NJ. Extend ON to point H so that ON = NH.

Step 2: Draw HJ and extend it to point A so that HJ – JA.

Step 3: Construct the circle with center O and radius OA.
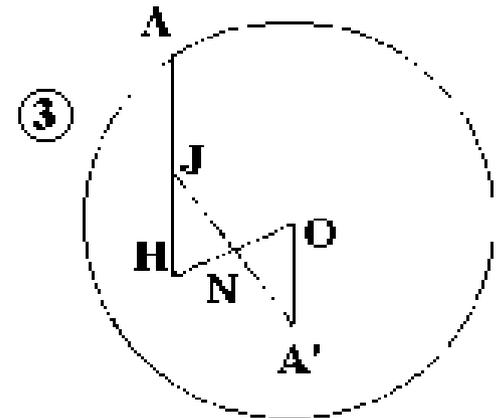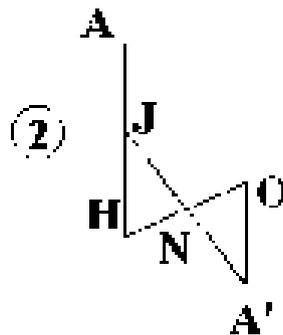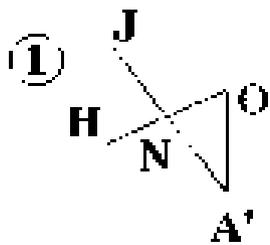
Step 4: Construct the perpendicular to OA' at A'. Extend this line until it meets the circle at B and C.

Constructions
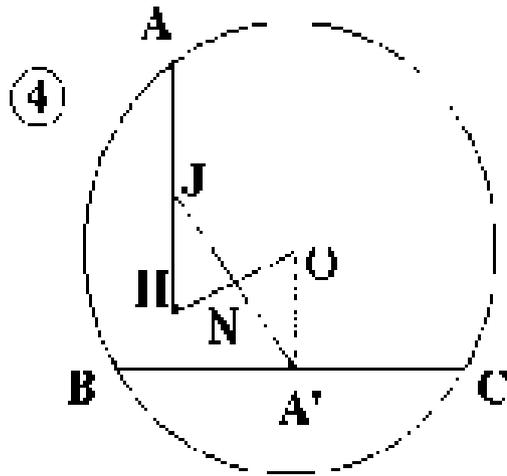
Step 5: Join A to B and C.
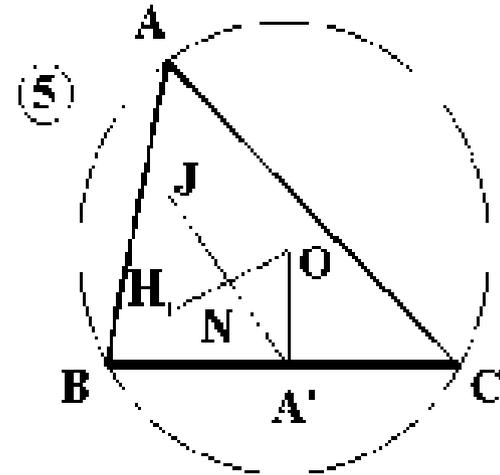
This construction always gives a unique triangle provided one exists. If N = A' there will be no nine-point circle, but N could equal O, or A' could equal O and the construction will still work. The points could also be collinear.

Impossibility Proofs

An algebraic analysis of the fields of constructible numbers shows the following:

Theorem: If a constructible number is a root of a cubic equation with rational coefficients, then the equation must have at least one rational root.

While we will not prove this result, we shall use it to investigate some old geometric problems that dealt with constructions.

Impossibility Proofs

The three famous problems of antiquity are:

The Delian problem - duplicating the cube. The problem is to construct a cube that has twice the volume of a given cube. A particular instance of this problem would be to construct a cube whose volume is twice that of the unit cube. This entails constructing a side of the larger cube, and in this case that means constructing a length equal to the cube root of 2. This length is a root of the equation $x^3 - 2 = 0$, but this cubic equation with rational coefficients has no rational root.

Impossibility Proofs

Trisection of an Angle - The problem is to find the angle trisectors for an arbitrary angle. The general problem can not be done because it can't be done for some specific angles, for instance an angle of 60º. (Construction of a 20 degree angle leads to the cubic equation $8x^3 - 6x - 1 = 0$, and this does not have roots of the required type). (Wankel)

Impossibility Proofs

Squaring the Circle - The problem is to construct a square that has the same area as the unit circle, i.e. π. If this can be done, then the square root of π would be constructible. And if that is true, then π would also be constructible. But π is a transcendental number (Lindemann, 1882), and such numbers are not constructible.

Angle Trisection

Angle Trisection can be done in many ways, some of which were known to the ancient Greeks. A simple method which uses a marked straightedge is due to Archimedes (287-212 B.C.) and another uses the **Conchoid of Nichomedes** (240 B.C.).

Trisect an arbitrary angle with a marked straightedge

O
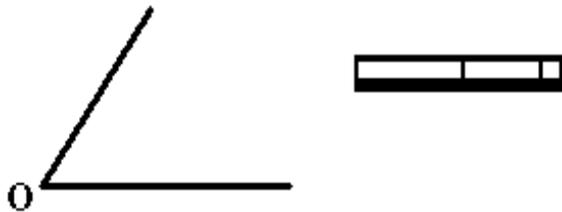
Step 1: On one side of the angle mark the point B so that OB is the length between the marks on the straightedge.

B

O
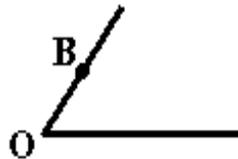
Step 2: Construct the circle with center B and radius OB.

B

O
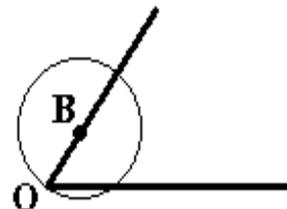
Archimedes' Angle Trisection

Step 3: Construct the parallel to OA that passes through B.
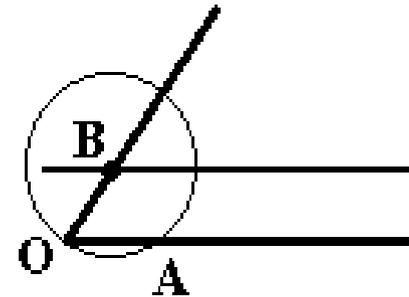
Step 4: Slide the straightedge so that it touches O and one mark is on the circle while the other is on the parallel line.
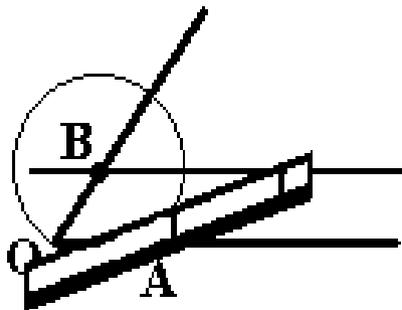
Step 5: Draw the line OT, it trisects the angle at O.
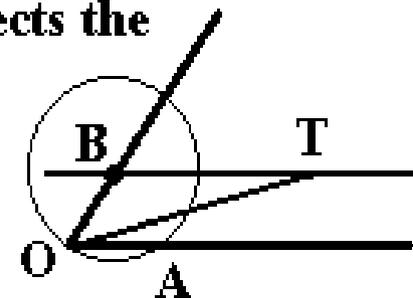
Archimedes' Angle Trisection
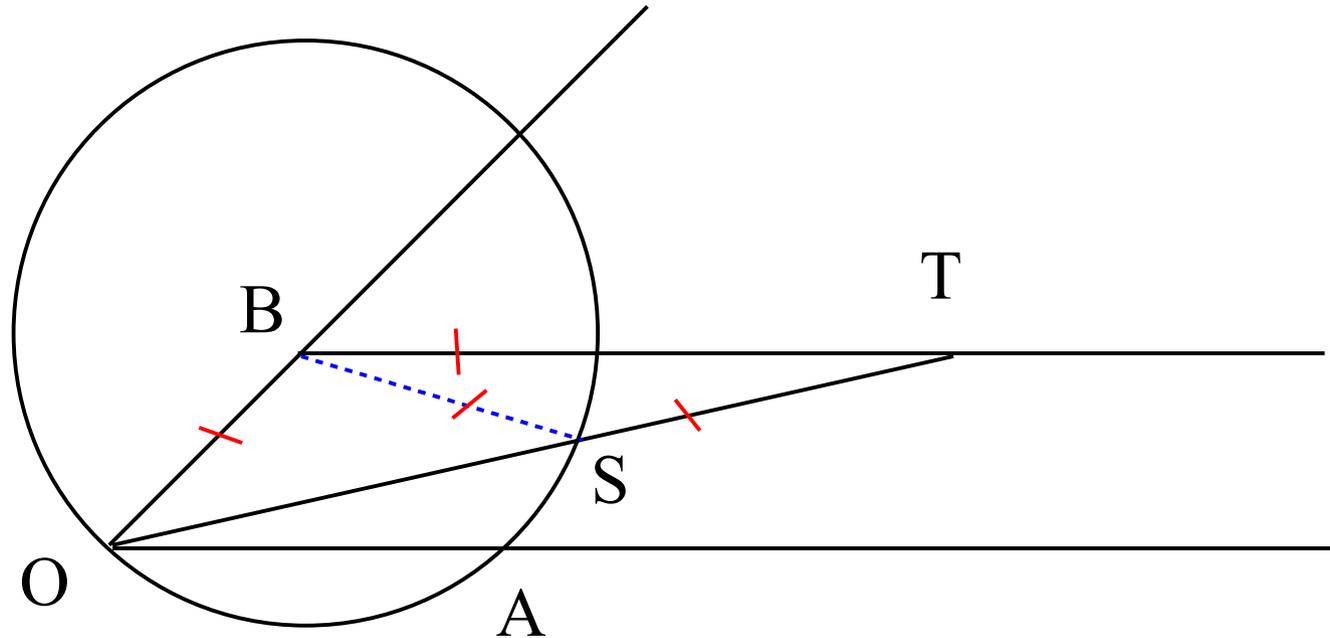
Let ∠AOT = x. ∠AOT ≅ ∠OTB (alternate interior angles of ∥ lines.) ∠OTB ≅ ∠TBS since ΔSBT is isosceles. ∠BSO = 2x since it is an exterior angle which is equal to the sum of the two opposite interior angles. ∠BOS ≅ ∠BSO since ΔBSO is isosceles. Therefore, ∠AOT is 1/3 of ∠AOB.

Conchoid of Nicomedes

Given a point O, a line *l* not through O and a length k we form the conchoid by adding the length k to all line segments drawn from O to *l*.
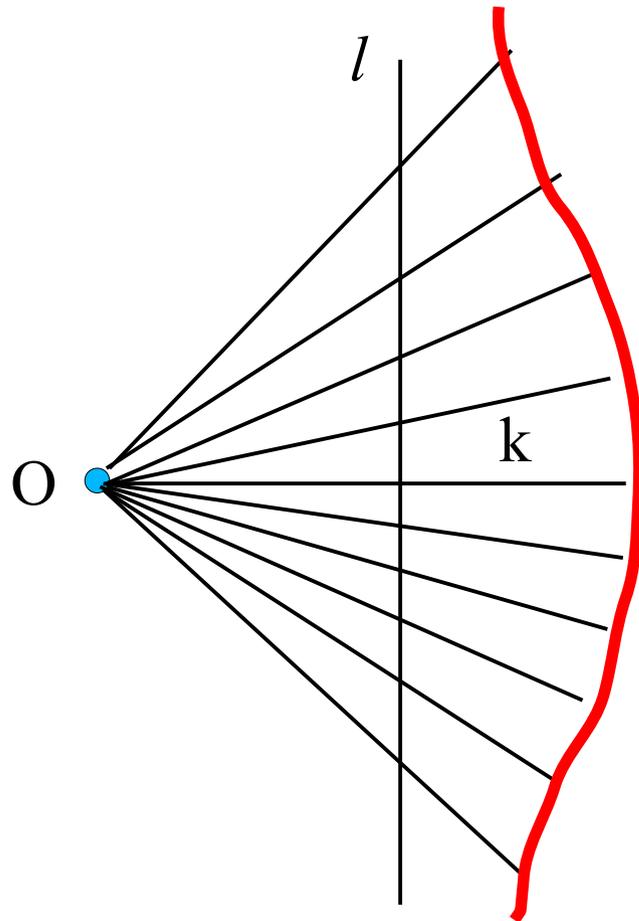
Conchoid of Nicomedes
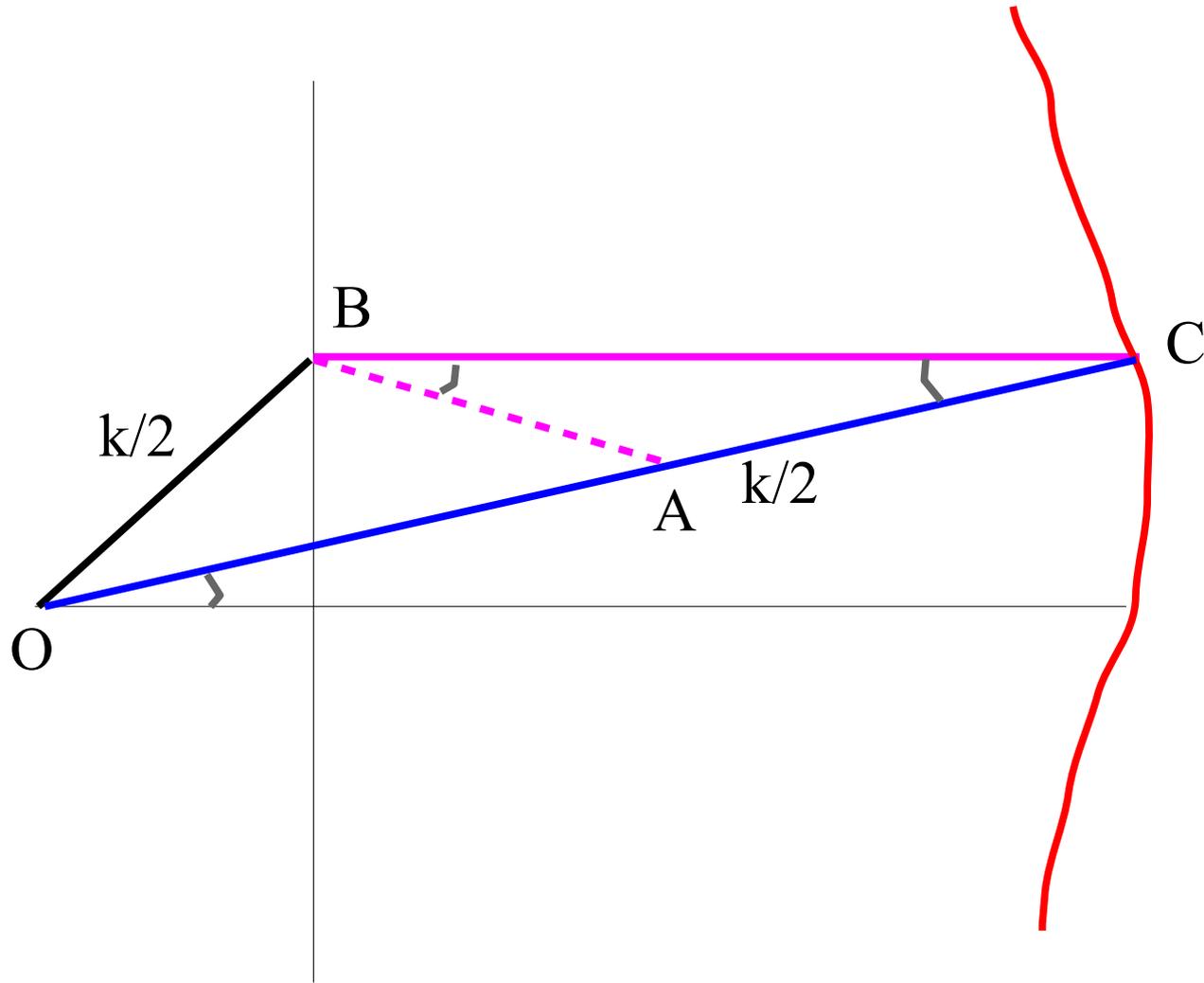
Circle Squarers

Circle squarers, angle trisectors, and cube duplicators are members of a curious social phenomenon that has plagued mathematicians since the earliest days of the science. They are generally older gentlemen who are mathematical amateurs (although some have had mathematical training) that upon hearing that something is impossible are driven by some inner compulsion to prove the authorities wrong.

Circle Squarers

In 1872, Augustus De Morgan's (1806-1871) widow edited and had published some notes that De Morgan had been preparing for a book, called *A Budget of Paradoxes*. A logician and teacher, De Morgan had been the first chair in mathematics of London University (from 1828). Besides his mathematical work, he wrote many reviews and expository articles and much on teaching mathematics. In the *Budget*, he examines his personal library and satirically barbs all the examples of weird and crackpot theories that he finds there. As he points out, these are just books that randomly came into his possession – he did not seek out any of this type of material. In the approximately 150 works he examined, there can be found 24 circle squarers and an additional 19 bogus values of π.

Angle Trisectors

DeMorgan's book was very successful. Today, with a couple of notable exceptions, there are hardly any circle squarers left.

However, their cousins, the Angle Trisectors are still with us.

Underwood Dudley, in 1987, wrote ***A Budget of Trisections*** in an attempt to do for Angle Trisectors what DeMorgan had done for Circle Squarers.

The comments and quotes that follow are all from Dudley's book.
(The 2nd edition came out in 1996 and was renamed ***The Trisectors***).

Angle Trisectors

There are several characteristics of angle trisectors (shared by others of their ilk) that may help you identify them.

1. *They are men*. Almost universally. Women seem to have more sense.
2. *They are old*. Often retired, having led a successful life in their chosen endeavors. Too much free time.
3. *They fail to understand what "impossible" means in mathematics*. The meaning is unfortunately not the same as the meaning in English. It is one of the great failures of mathematics education that this essential difference is not made plain to students.

Angle Trisectors

Typical is the trisector who wrote

"I received through the mail an advertising brochure, from a science magazine, that had in it a simple statement – and it went something like this – the FORMULA for TRISECTING AN ANGLE had never been worked out. This really intrigued me. I couldn't believe, after hundreds of years of math, that this could be true."

So he went to the library and found that all the books agreed that it was impossible.

"How could men of science be so stupid? Any scientist or mathematician who declares that a thing is impossible is *showing his limitations* before he even starts on the problem at hand."

Angle Trisectors

Another trisector wrote in 1933:

"Moreover, we find our modern authorities of mathematics not attempting to solve these unsolved problems, but writing treatises showing the impossibility of proving them. Instead of offering inducements to the solution of these problems, they discourage others and dub them as 'cranks'. "

4. *They do not know much mathematics*. Often, high school is the last place they have seen any formal mathematics.

"It was necessary to get outside of the problem to solve it, and it was not solved by a study of geometry and trigonometry, as the author has never made a study of these branches of learning."

Angle Trisectors

5. *They think the problem is important*. Since Archimedes work, there has not been any need for such a construction, yet they persist in thinking that mathematics has been stymied by this lack.

"It having been hitherto deemed impossible to geometrically trisect or divide any angle into any number of equal parts, or fractions of parts, the author of the present work has devoted careful study to the solving of the problem so useful and necessary to every branch of science and art, that requires the use of geometry."

"The study of technical magazines and data shows that a solution is being sought whereby a standard construction permits the thrice division of any given angle …"

Angle Trisectors

6. *They believe that they will be richly rewarded for their work*. No one has ever put up a prize for a solution.

"When the time came for me to submit this project to a publisher, I was very much concerned about the copyright. I was fearful that if I submitted to a publisher, they might steal the entire trisection and I would have to go to court and try to establish my right to the trisection."

Angle Trisectors

7. *They are not logical*. For instance,

"Those who are skeptical should offer something more than rhetoric or argument in order to disprove geometrical *facts.* Assuming the angle and its trisectors given, the enveloping quadrantal arc constructed, and its points of trisection found, if it be denied that the trisectors pass through these points of equal division on the quadrantal arc, let them show by *the ruler and compasses* where these lines and points *are* with respect to each other on the quadrant. If the lines constituting the respective pairs of trisectors of both sectors do not intersect on the quadrantal arc they should show by *the ruler and compasses where they do intersect.*"

To prove him wrong you have to trisect an angle with ruler and compass. !!!!

Angle Trisectors

8. *They are loners*. They work by themselves, sometimes using books, but never discuss their work until it is completed. Even though they do not communicate with each other, they do tend to swarm.

For instance, in 1754, Jean Étienne Montucla, an early French historian of mathematics, wrote a legitimate history of the quadrature problem. A year later, the French Academy of Sciences was forced to publicly announce that it would no longer examine any solutions of the quadrature problem.

Angle Trisectors

9. *They are prolific writers*. Here is what De Morgan says about Milan whose method gave $\pi = 3.2$ in 1855:

[The circle-squarer] is active and able, with nothing wrong with him except his paradoxes. In the second tract named he has given the testimonials of crowned heads and ministers, etc. as follows. Louis Napoleon gives thanks. The minister at Turin refers it to the Academy of Sciences and hopes so much labor will be judged worthy of esteem. The Vice-Chancellor of Oxford – a blunt Englishman – begs to say that the University has never proposed the problem, as some affirm. The Prince Regent of Baden has received the work with lively interest. The Academy of Vienna is not in a position to enter into the question. The Academy of Turin offers the most *distinct* thanks. The Academy della Crusca attends only to literature, but gives thanks. The Queen of Spain has received the work with the highest appreciation. The University of Salamanca gives infinite thanks, and feels true satisfaction in having the book.

Angle Trisectors

Lord Palmerston gives thanks. The Viceroy of Egypt, not yet being up in Italian, will spend his first moments of leisure in studying the book, when it shall have been translated into French: in the mean time he congratulates the author upon his victory over a problem so long held insoluble. All this is seriously published as a rate in aid of demonstration. If those royal compliments cannot make the circumference about 2 per cent larger than geometry will have it – which is all that is wanted – no wonder that thrones are shaky.
- *Budget*, Vol. 2, pp. 61-2.

Angle Trisectors

Now, will you know a Angle Trisector when you see one coming? And will you know what to do?

Hint: What you do involves your legs.

No, you do **not** kick him!

Regular Polygons (Gauss)

These are only possible when the number of sides, n, is of the form

$$n = 2^a p_1 p_2 ... p_k$$

where the p_i are distinct Fermat primes, i.e. prime numbers of the form

$$p_{i+1} := 2^{2^i} + 1$$

The first few Fermat primes are: $p_1 = 3$, $p_2 = 5$, $p_3 = 17$.

Thus, it is possible to construct regular polygons of n sides when n is: 3, $4 = 2^2$, 5, $6 = 2(3)$, $8 = 2^3$, $10 = 2(5)$, $12 = 2^2(3)$, $15 = 3(5)$, $16 = 2^4$ and 17.

Regular Polygons (Gauss)

The factor of a power of 2 comes from the fact that given any regular n-gon, you can always construct a regular 2n-gon. This is done by inscribing the n-gon in a circle and then constructing the perpendicular bisectors of each of the sides. Extend these to the circle and these points together with the original vertices of the n-gon, form the vertices of a regular 2n-gon. Repeating this will give the higher powers of 2.

It is not possible to construct, **with straightedge and compass alone**, regular polygons of sides n = 7, 9, 11, 13, 14, 18, 19, ...

Other types of Constructions

It can be shown that any construction that can be made with straightedge and compass can be made with compass alone (Mascheroni, 1797 [Mohr, 1672]). Of course one must understand that a straight line is given as soon as two points on it are determined, since one can't actually draw a straight line with only a compass.

It can also be shown that any construction that can be made with straightedge and compass can be made with straightedge alone, as long as there is a single circle with its center given (Steiner, 1833 [Poncelet, 1822]).

Paper Folding

Folding the perpendicular bisector of a line segment
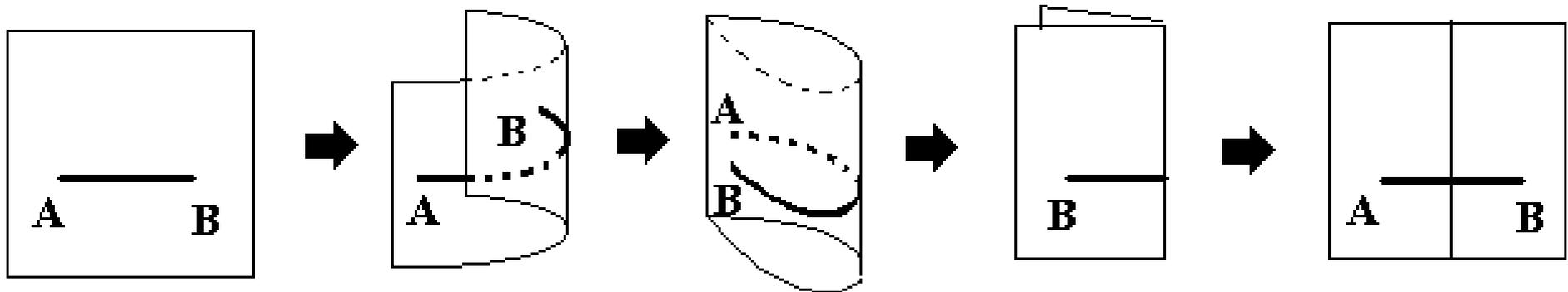
Folding the angle bisector of a given line
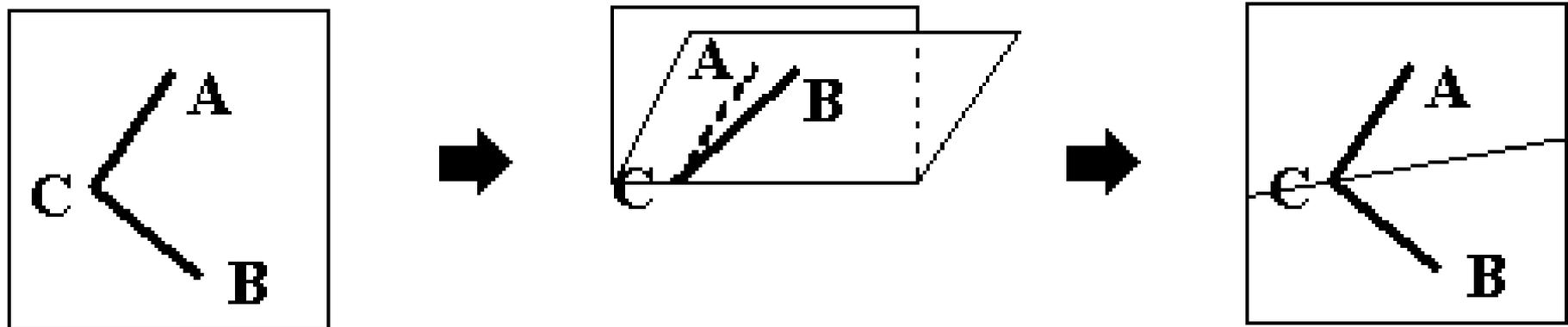
Paper Folding

Fold a perpendicular from a point to a line
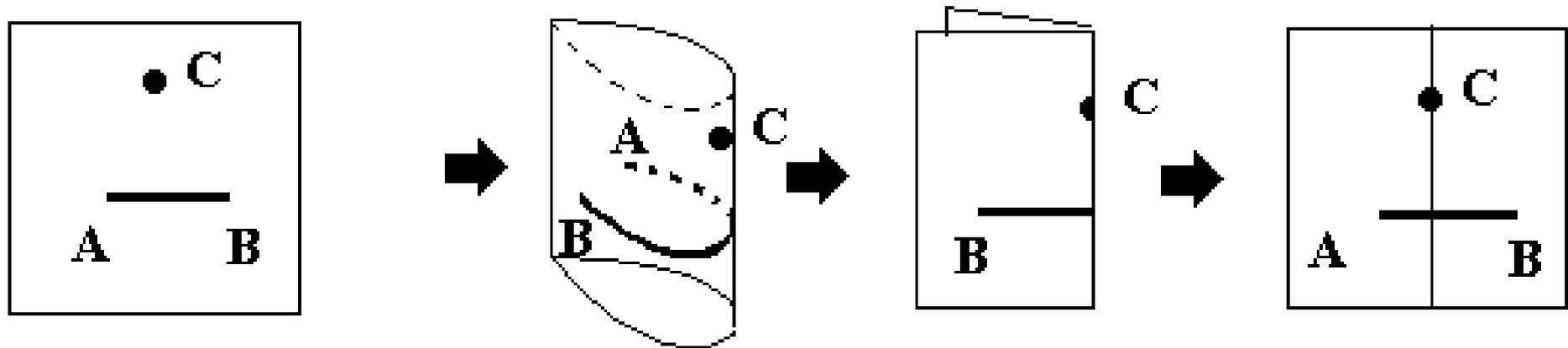
Fold a parallel to a given line through a point not on that line
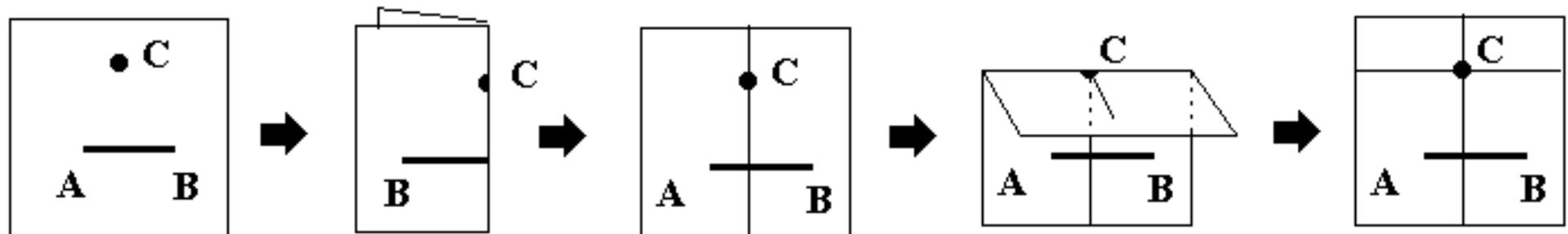
Paper Folding

Other constructions are possible, including trisecting angles.

Here is a folding of the regular nonagon (9 sided regular polygon) which is impossible to do with straightedge and compass (from T.S. Row, *Geometric Exercises in Paper Folding*).

Unfortunately, this is only approximate, this construction can not be done exactly with paper folding.