Equivalence Relations

Definition

An ***equivalence relation*** on a set S, is a relation on S which is *reflexive, symmetric* and *transitive*.

Examples:

Let S = $\mathbb{Z}$ and define R = {(x,y) | x and y have the same parity} i.e., x and y are either both even or both odd.

The parity relation is an equivalence relation.

1. For any x ∈ $\mathbb{Z}$, x has the same parity as itself, so (x,x) ∈ R.
2. If (x,y) ∈ R, x and y have the same parity, so (y,x) ∈ R.
3. If (x,y) ∈ R, and (y,z) ∈ R, then x and z have the same parity as y, so they have the same parity as each other (if y is odd, both x and z are odd; if y is even, both x and z are even), thus (x,z) ∈ R.

Examples

Let S = $\mathbb{R}$ and define the "square" relation R = $\{(x,y) \mid x^2 = y^2\}$.

The square relation is an equivalence relation.

1. For all $x \in \mathbb{R}$, $x^2 = x^2$, so $(x,x) \in R$.
2. If $(x,y) \in R$, $x^2 = y^2$, so $y^2 = x^2$ and $(y,x) \in R$.
3. If $(x,y) \in R$ and $(y,z) \in R$ then $x^2 = y^2 = z^2$, so $(x,z) \in R$.

For any set S, the identity relation on S, $I_S = \{(x,x) \mid x \in S\}$.

This is an equivalence relation for rather trivial reasons.

1. obvious
2. If $(x,y) \in R$ then $y = x$, so $(y,x) = (x,x) \in R$.
3. If $(x,y) \in R$ and $(y,z) \in R$ then $x = y = z$, so $(x,z) = (x,x) \in R$.

Modular Arithmetic

Let $S = \mathbb{Z}$. For each positive integer m, we define the ***modular relation*** $\equiv_m$,by $x \equiv_m y$ iff $m \mid (x-y)$, i.e. $\equiv_m = \{(x,y) : m \mid x - y\}$.

Examples:

$7 \equiv_5 2$, $11 \equiv_5 1$, $10 \equiv_5 0$, $-12 \equiv_5 3$

$7 \equiv_3 1$, $11 \equiv_3 2$, $10 \equiv_3 1$, $-12 \equiv_3 0$

Another way to think about the modular relation is:

$x \equiv_m y$ iff x and y have the same remainder when divided by m.

By the division algorithm, $x = mq_1 + r_1$, $y = mq_2 + r_2$, so $x - y = m(q_1-q_2) + (r_1-r_2)$ so, $m \mid x-y$ iff $m \mid r_1 - r_2$. Since $|r_1-r_2| < m$, $m \mid r_1 - r_2$ iff $r_1 - r_2 = 0$ iff $r_1 = r_2$.

Modular Arithmetic

Theorem: For any natural number m, the modular relation $\equiv_m$ is an equivalence relation on $\mathbb{Z}$.

Pf: For any x in $\mathbb{Z}$, since $x - x = 0$ and $m \mid 0$, $x \equiv_m x$. (Reflexitivity)

If $x \equiv_m y$ then $m \mid x - y$. Since $y - x = -(x-y)$, $m \mid y - x$, and so, $y \equiv_m x$. (Symmetry)

If $x \equiv_m y$ and $y \equiv_m z$ then $m \mid x - y$ and $m \mid y - z$. Since

$$x - z = (x - y) + (y - z)$$

we have $m \mid x - z$, so $x \equiv_m z$. (Transitivity)

Equivalence Classes

Given an equivalence relation R on a set S, we define the ***equivalence class containing an element x*** of S by:

$$[x]_R = \{y \mid (x,y) \in R\} = \{y \mid x \; R \; y\}.$$

The text uses the notation x/R (which I am not fond of) for what I have called $[x]_R$.

Examples:

Let S = {1, 2, 3} and R = {(1,1), (2,2), (3,3), (1,2), (2,1)}.
Then [1] = {1,2}   [2] = {1,2}   [3] = {3}.

Let $S = \mathbb{R}$ and $R = \{(x,y) \mid x^2 = y^2\}$.
Then [0] = {0}, [1] = {1,-1}, [¼] = {**¼, -¼**}, [x] = {x, -x}.

Equivalence Classes

More Examples:

Let $S = \mathbb{Z}$ and $R = \{(x,y) \mid x \text{ and } y \text{ have the same parity}\}$.

$[0] = [2] = ... = [2k] = \{0, \pm 2, \pm 4, \pm 6, ..., \pm 2k, ...\}$

$[-1] = [1] = ... = [2k+1] = \{\pm 1, \pm 3, \pm 5, ..., \pm 2k+1, ...\}$

For any set S, $I_S = \{(x,x) \mid x \in S\}$.

$[a] = \{a\}$ for all $a \in S$.

Let $S = \mathbb{Z}$ and $R =$ " $\equiv_5$ ".

$[0] = \{0, \pm 5, \pm 10, \pm 15, ..., 5k\}$ $(k \in \mathbb{Z})$

$[1] = \{..., -9, -4, 1, 6, 11, ..., 5k + 1\}$

$[2] = \{..., -8, -3, 2,7,12, ..., 5k+2\}$

$[3] = \{..., -7, -2, 3, 8, 13, ..., 5k+3\}$

$[4] = \{..., -6, -1, 4, 9, 14, ..., 5k+4\}$

Properties of Equivalence Classes

Let R be an equivalence relation on the set S.

I. ***For all $x \in S$, $x \in [x]$.***

Since R is reflexive, $(x,x) \in R$ for all $x \in S$.

II. ***If $y \in [x]$ then $x \in [y]$, and $[x] = [y]$.***

Since R is symmetric, if $y \in [x]$ then $(x,y) \in R$ so $(y,x) \in R$ and we have $x \in [y]$. If $s \in [x]$, then $(x,s) \in R$, so $(s,x) \in R$ and then $(s,y) \in R$ (by transitivity) and finally $(y,s) \in R$, so $s \in [y]$. Similarly, if $t \in [y]$ then $t \in [x]$ and so, $[x] = [y]$.

III. ***For any x and $y \in S$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.***

If there is a $z \in [x]$ which is also in $[y]$, then $(x,z) \in R$ and $(y,z) \in R$. By symmetry, $(z,y) \in R$ as well. By transitivity, $(x,y) \in R$, so $y \in [x]$. By II, $[x] = [y]$.

An Important Equivalence Relation

Let S be the set of fractions:

$$S = \left\{ \frac{p}{q} :\ p, q \in \mathbb{Z}, q \neq 0 \right\}$$

Define a relation R on S by:

$$\frac{a}{b}\ R\ \frac{c}{d} \quad \text{iff} \quad ad = bc.$$

This relation is an equivalence relation.

1) For any fraction a/b, a/b R a/b since ab = ba. (Reflexitivity)
2) If a/b R c/d, then ad = bc, so cb = da and c/d R a/b. (Symmetry)
3) If a/b R c/d, and c/d R e/f, then ad = bc and cf = de. Multiply the first equation by f, to get adf = bcf , so adf = bde. Divide by d (which is not 0) to get af = be, so a/b R e/f. (Transitivity)

An Important Equivalence Relation

The equivalence classes of this equivalence relation, for example:

$$\left[\frac{1}{1}\right]=\left\{\frac{2}{2},\frac{3}{3},\cdots,\frac{k}{k},\cdots\right\}$$

$$\left[\frac{1}{2}\right]=\left\{\frac{2}{4},\frac{3}{6},\frac{4}{8},\cdots,\frac{k}{2k},\cdots\right\}$$

$$\left[\frac{4}{5}\right]=\left\{\frac{4}{5},\frac{8}{10},\frac{12}{15},\cdots,4\frac{k}{5}k,\cdots,\right\}$$

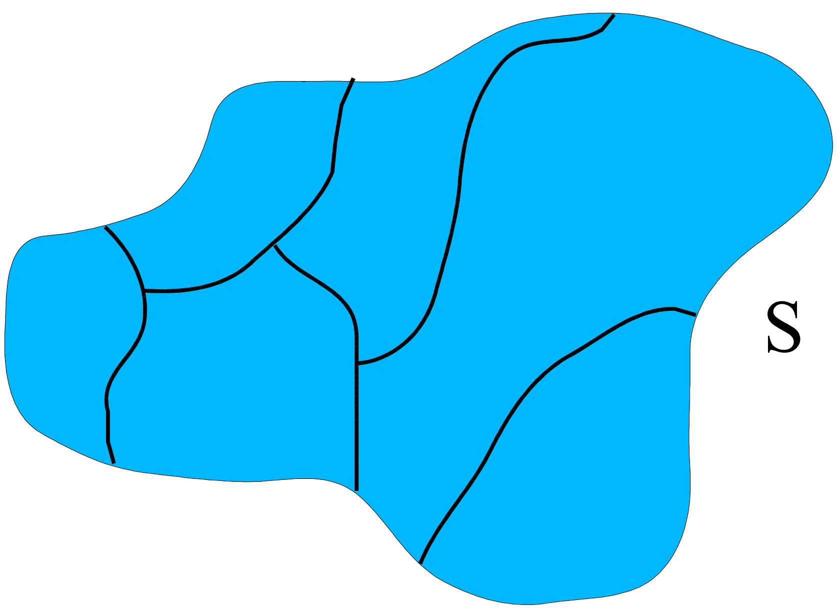
are called ***rational numbers***. The set of all the equivalence classes is denoted by $\mathbb{Q}$.

Partitions

A ***partition*** of a set S is a family *F* of *non-empty* subsets of S such that

(i) if A and B are in *F* then either A = B or $A \cap B = \varnothing$, and

(ii) $\underset{A \in F}{union}\, A = S$.



Partitions

If S is a set with an equivalence relation R, then it is easy to see that the equivalence classes of R form a partition of the set S.

More interesting is the fact that the converse of this statement is true.

Theorem 3.6 : Let *F* be any partition of the set S. Define a relation on S by x R y iff there is a set in *F* which contains both x and y. Then R is an equivalence relation and the equivalence classes of R are the sets of *F*.

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Let *F* be any partition of the set S. Define a relation on S by x R y iff there is a set in *F* which contains both x and y. Then R is an equivalence relation and the equivalence classes of R are the sets of *F*.

Pf: Since F is a partition, for each x in S there is one (and only one) set of F which contains x. Thus, x R x for each x in S (R is reflexive)
If there is a set containing x and y then x R y and y R x both hold. (R is symmetric).
If x R y and y R z, then there is a set of F containing x and y, and a set containing y and z. Since F is a partition, and these two sets both contain y, they must be the same set. Thus, x and z are both in this set and x R z (R is transitive).
Thus, R is an equivalence relation.

Theorem 3.6

Consider the equivalence classes of this equivalence relation.

$[x] = \{y \mid x \text{ and } y \text{ are in some set of } F\}$.

Let A be a set of the partition *F*. Since A is non-empty, it contains an element x.

Now, $y \in A$ iff $y \in [x]$, so $A = [x]$.

Order Relations

Partial Orders

Definition: A relation R on a set A is a ***partial order*** (or ***partial ordering***) for A if R is *reflexive, antisymmetric* and *transitive*.
A set A with a partial order is called a ***partially ordered set***, or ***poset.***

Examples:
The natural ordering " $\leq$ "on the set of real numbers $\mathbb{R}$.

For any set A, the subset relation $\subseteq$ defined on the power set *P* (A).

Integer division on the set of natural numbers $\mathbb{N}$.

Predecessors

Definiton: Let R be a partial ordering on a set A and let a,b ∈ A with a ≠ b. Then a is an ***immediate predecessor*** of b if a R b and there does not exist c∈ A such that c ≠ a, c ≠ b, a R c and c R b.

Examples:

Consider the partial order "≤" on ℤ.

5 is an immediate predecessor of 6 since 5 ≤ 6 and there is no integer c not equal to 5 or 6 which satisfies 5 ≤ c ≤ 6.

3 is not an immediate predecessor of 6 since 3 ≤ c ≤ 6 is satisfied by 4 or 5.

Now consider the partial order given by integer division on ℤ.

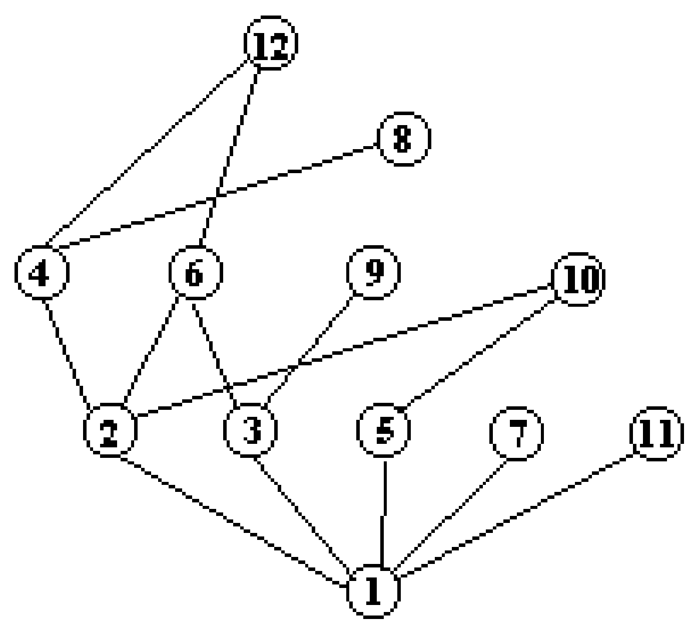
3 is an immediate predecessor of 6 since there is no integer c which 3 divides and which divides 6 other than 3 or 6.

3 is also an immediate predecessor of 9, but not of 12.

Hasse Diagrams

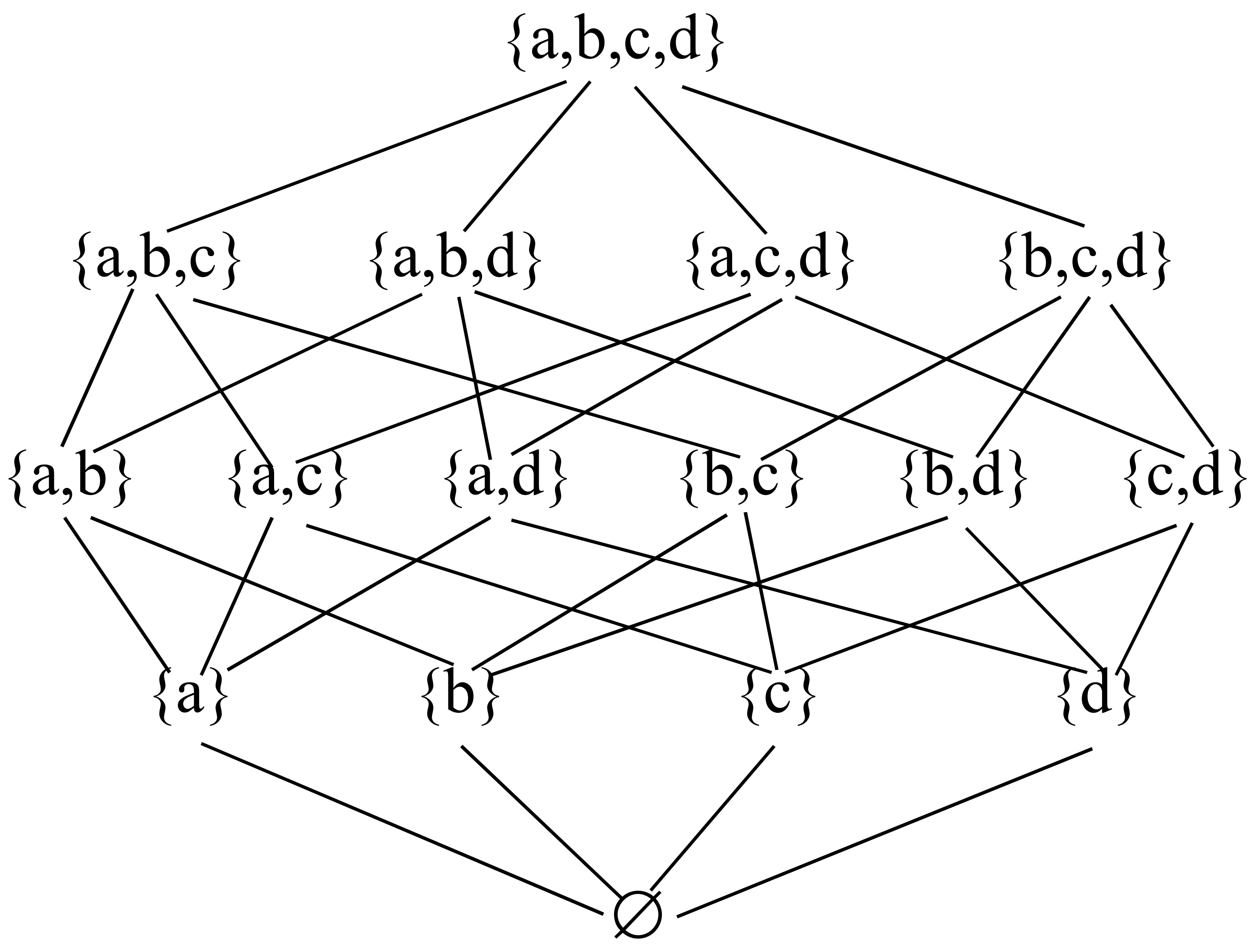
Definition: A ***Hasse diagram*** for a partial order is a digraph representing this relation in which only the arcs to immediate predecessors are drawn and the digraph is drawn so that all arcs are directed upwards (we then remove the arrow heads).

Example: Consider the poset {1,2, ...,12} with integer division as the partial order. The Hasse diagram for this poset is given by



Hasse Diagrams

Let S = {a,b,c,d} and consider *P*(S) with subset inclusion (a poset). The Hasse diagram would be:



Supremum

Definition: Let R be a partial order for A and let B be any subset of A. Then a∈ A is an ***upper bound*** for B if for every b ∈ B, b R a. Also, a is called a ***least upper bound*** (or ***supremum***) for B if

1) a is an upper bound for B, and
2) a R x for every upper bound x for B.

Example: Reals with the usual ordering.

$B = \{x \mid 5 < x < 7\}$

8, 7.5, 11, 7.00001 are all upper bounds of B.

7 is also an upper bound, and 7 ≤ any of the upper bounds, so 7 is a least upper bound of B (Note that 7 is not in B).

Infimum

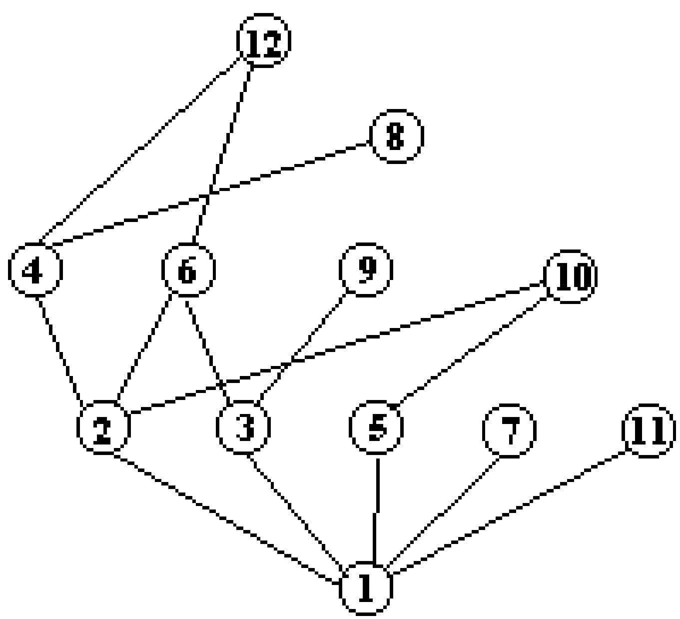
Similarly, $a \in A$ is a ***lower bound*** for B if for every $b \in B$, a R b.
Also, a is called a ***greatest lower bound*** (or ***infimum***) for B if

1) a is a lower bound for B, and
2) x R a for every lower bound x for B.

We write ***sup(B)*** [sometimes l.u.b.(B)] to denote the supremum of B and ***inf(B)*** [sometimes g.l.b.(B)] to denote the infimum for B.

Example

In the example below (Hasse diagram), consider B = {2,3}. Both 6 and 12 are upper bounds of this subset and 6 is the sup(B).
The only lower bound of B is 1 and inf(B) = 1.
Now let B = {4,6}. Sup{B) = 12 and Inf(B) = 2.
Consider the set C = {2,3,5}. There is no upper bound for C, and 1 is a lower bound and also inf(C).



Theorem 3.8

If R is a partial order for a set A, and B $\subseteq$ A, then if sup(B) (or inf(B)) exists, it is unique.

Pf: Suppose that c and d are both supremums for the subset B.
c and d are upper bounds of B.
Since c is a sup(B) and d is an upper bound, c R d.
Since d is a sup(B) and c is an upper bound, d R c.
R is antisymmetric, so, c R d and d R c $\Rightarrow$ c = d.
Thus, sup(B), if it exists, is unique.

The argument for inf(B) is similar.

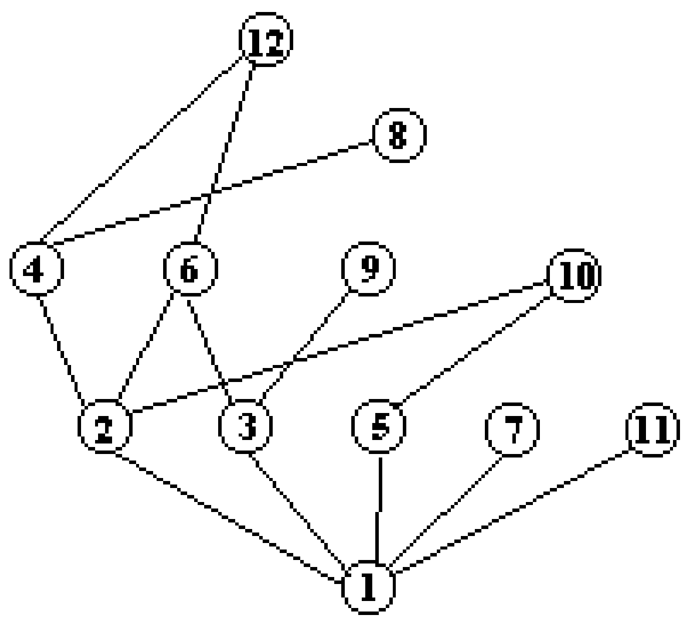
Largest and Smallest Elements

Definition: Let R be a partial order for a set A. Let B $\subseteq$ A. If inf (B) exists *and is an element of B*, it is called the ***smallest*** (or least) ***element*** of B.

If sup(B) exists *and is an element of B*, it is called the ***largest*** (or greatest) ***element*** of B.

Largest and Smallest Elements

Examples: Continuing with our example, we have the subset B = {2,3}. B has no smallest element since inf(B) = 1 and 1 is not in B. On the other hand, the set D = {2,4,6,12} has both a largest element, 12 (since sup(D) = 12) and a smallest element, 2 (since inf(D) = 2). Notice that {2,4,6} would have a smallest element but not a largest element.

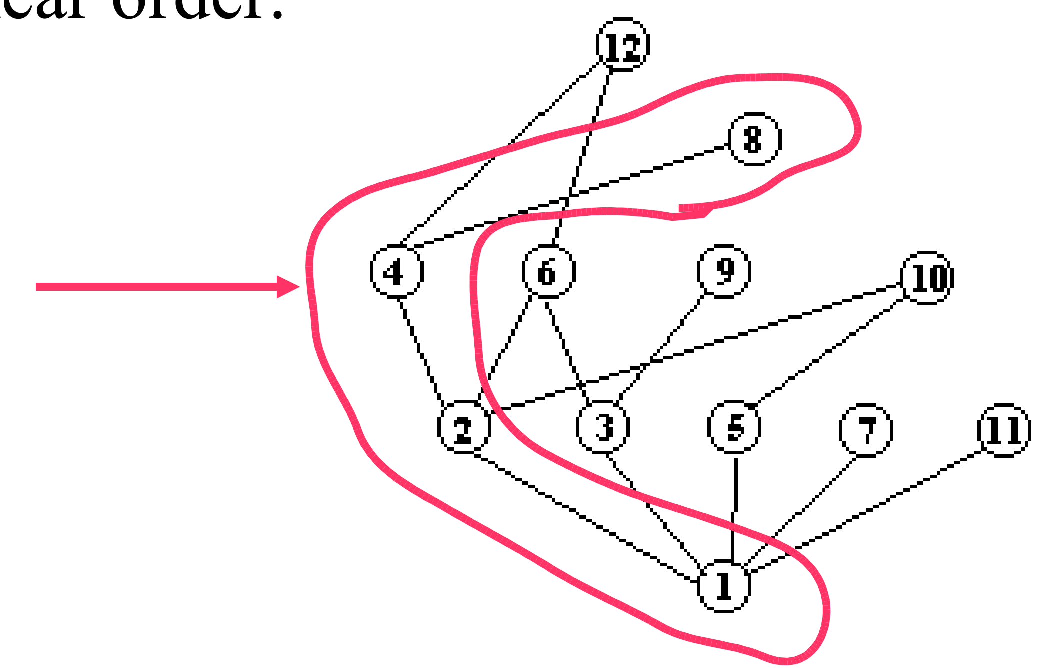


Total Orders

Definition: A partial ordering R on a set A is called a ***linear order*** (or ***total order***) on A if for any two elements x and y of A, either x R y or y R x. (Sometimes we say that every two elements are *comparable*.)

Examples: Our example is not a total order since, for example, 2 and 11 are not comparable. However, if we restrict the set to {1,2,4,8} then we do have a linear order.

Subset with a total order



Total Orders

As for our other two examples:

The natural ordering " $\leq$ "on the set of real numbers $\mathbb{R}$ is a total order since for any two real numbers x and y either $x \leq y$ or $y \leq x$.

For any set A, the subset relation $\subseteq$ defined on the power set P (A) is not a total order if A has at least two elements since if a and b in A, then $\{a\} \not\subseteq \{b\}$ and $\{b\} \not\subseteq \{a\}$.

Well Ordering

Definition: Let L be a linear ordering on a set A. L is a ***well ordering*** on A if every nonempty subset B of A contains a smallest element.

Examples: We know that the natural numbers, $\mathbb{N}$ with the usual ordering, $\leq$ is a well ordered set.

Any totally ordered **finite** set is a well ordered set.

The integers, $\mathbb{Z}$ with the usual ordering is not a well ordered set, but if you introduce a different ordering on this set, for instance, use the partial order given by {0,-1,1,-2,2,-3,3,....} then we do get a well ordered set.

Well Ordering

There is a theorem, proved in an advanced course in set theory, that says that any set can be well ordered. That is, given an arbitrary set S, one can find a total order on S which is a well ordering.

The theorem is an existence theorem, proved by contradiction, and does not give a construction of a well ordering that it proves exists.

The theorem says that there is a well ordering of $\mathbb{R}$ (obviously not the usual ordering), but no one knows how to construct such a well ordering!!