Completeness of the Reals: a synopsis
Bounds Again

Let $A$ be a subset of an ordered field $F$. We say that $u \in F$ is an **upper bound** for $A$ iff $a \leq u$ for all $a \in A$. If $A$ has an upper bound, $A$ is **bounded from above**.

Likewise, $l \in F$ is a **lower bound** for $A$ iff $l \leq a$ for all $a \in A$. $A$ is **bounded from below** if any lower bound for $A$ exists.

The set $A$ is **bounded** iff $A$ is both bounded from above and bounded from below.
Complete Fields

In some fields bounded sets may not have sup's or inf's. For example in \( \mathbb{Q} \), the set \{x \mid x^2 < 2\} is bounded but does not have a least upper bound nor a greatest lower bound in \( \mathbb{Q} \).

This set is the set of rational numbers \( x \) where \( -\sqrt{2} < x < \sqrt{2} \)

This set is bounded above by 3 and bounded below by -3, so it is a bounded set. However, there is no rational number which is, for instance, an upper bound which is smaller than every upper bound of the set, i.e., no least upper bound.

An ordered field \( F \) is **complete** iff every nonempty subset of \( F \) that has an upper bound in \( F \) has a least upper bound (supremum) that is in \( F \).
The Reals

The rationals are not complete, but the reals form a complete ordered field. The completeness property is sometimes referred to as the Least Upper Bound Property.

**Theorem:** In a complete ordered field, every non-empty set which has a lower bound has a greatest lower bound.

**Pf:** Let $A$ be a set with a lower bound. Consider the set $B$ of all the lower bounds of $A$. $B$ is non-empty, and any element of $A$ is an upper bound of $B$. By completeness, $B$ has a least upper bound. Let $b = \sup(B)$. Now $b$ is a lower bound of $A$ (if not there is an element $a$ of $A$ so that $a < b$ and so, $a$ is a smaller upper bound of $B$) and if $c$ is any lower bound of $A$, then $c \leq b$ (since $b$ is the sup of $B$).
Density of the rationals

**Theorem:** Between any two real numbers there is a rational number.

**Proof:** Let $a$ and $b$ be real numbers with $a < b$. Let $q$ be a positive integer for which $1/q < b - a$. The set

$$P = \{p/q : p \text{ an integer}\}$$

has no bounds, either upper or lower. In particular, $a$ is not an upper bound for $P$ (nor lower bound). Thus, there exists an integer $p^*$ so that $p/q \leq a$ for $p \leq p^*$ and $p/q > a$ for $p > p^*$. Then the rational number

$$r = (1 + p^*)/q$$

is greater than $a$ and less than $b$, so it is a rational number between $a$ and $b$. 
Density of the rationals

This theorem also shows that there are rational numbers within any prescribed positive distance of a given real number. For a real number $a$ and distance $\varepsilon$ there must be a rational number between $a - \varepsilon$ and $a + \varepsilon$. Such a rational number will be at distance less than $\varepsilon$ from $a$.

The fact that there are rational numbers arbitrarily close to every real number is expressed by saying that the set of rational numbers is \textit{dense} in $\mathbb{R}$. 
Completeness of \( \mathbb{IR} \)

It can be shown that any complete ordered field is just a copy (with the elements renamed) of the reals, so in this sense there is only one complete ordered field.

Since this property is not as familiar as the algebraic properties of the real numbers, we should investigate it more carefully.
Distance

The distance between two real numbers $a$ and $b$ is defined to be the absolute value of their difference, $d(a,b) = |a - b|$.

This distance function has the following properties:

1) $d(a,b) \geq 0$ and $d(a,b) = 0$ only if $a = b$;
2) $d(a,b) = d(b,a)$; and
3) $d(a,c) \leq d(a,b) + d(b,c)$ for any real numbers $a$, $b$ and $c$.

Let $a$ be a real number and $B$ a non-empty set of $\mathbb{R}$. The distance from $a$ to $B$, $d(a,B)$ is the inf (greatest lower bound) of the set of distances $d(a,b)$ for all $b$ in $B$.

$$d(a,B) = \inf \{|a-b| : b \in B\}.$$
Neighborhoods

For any real number $a$, if $\delta$ is a positive real number, the \textit{$\delta$-neighborhood of $a$} is the set

$$N(a, \delta) = \{x \in \mathbb{R}: |x - a| < \delta\} = (a - \delta, a + \delta).$$

That is $N(a, \delta)$ is the open interval centered at $a$ with length $2\delta$.

So, $N(5, .03)$ is the open interval $(4.97, 5.03)$. 
Open Sets

For a set $A \subseteq \mathbb{R}$, a point $x$ is an **interior point of $A$** iff there exists a $\delta > 0$ such that $N(a, \delta) \subseteq A$. The set $A$ is **open** iff every point of $A$ is an interior point of $A$.

**Example**: Any open interval is an open set.
Open Sets

The following statements are equivalent for a subset $S$ of $\mathbb{R}$:

(a) $S$ is an open set.

(b) $S$ is the union of a family of open intervals.

(c) When $S \neq \mathbb{R}$, for each $x \in S$, $d(x, \mathbb{R}\setminus S) > 0$.

Proof technique: In proving TFAE (the following are equivalent) statements we arrange the statements in a cycle (with possible side branches) and then only prove the statements following the cycle. In this simple case, we would prove: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$
Proof

(a) \Rightarrow (b) Each point x of an open set A is contained in an open neighborhood N(x,\delta), which is an open interval, contained in A. Fix one of these open neighborhoods for each x. The union of the selected neighborhoods is A.

(b) \Rightarrow (c) Postponed.

(c) \Rightarrow (a) For each x in set A, let \delta = d(x, \mathbb{R}\setminus A) > 0. The neighborhood N(x,\delta/2) is an open interval contained in A and containing x. Thus, A is open.
Open Sets

Properties of open sets:

(a) \( \mathbb{R} \) and \( \emptyset \) are open sets.

(b) If \( \{O_\alpha : \alpha \in A\} \) is a collection of open sets, then
\[
\bigcup_{\alpha \in A} O_\alpha \text{ is open.}
\]

(c) If \( \{O_i : 1 \leq i \leq k\} \) is a finite collection of open sets, then
\[
\bigcap_i O_i \text{ is open.}
\]

(d) A function \( f: \mathbb{R} \to \mathbb{R} \) is \textit{continuous} iff \( f^{-1}(O) \) is an open set for all open sets \( O \).
Closed Sets

The set $A$ is **closed** iff its complement is open.

**Example:** Any closed interval is a closed set.

A set of reals need not be either open nor closed. Some sets can be both open and closed (**clopen sets**).
Closed Sets

The following are properties of closed sets:

(a) \( \mathbb{R} \) and \( \varnothing \) are closed sets.

(b) If \( \{C_\alpha : \alpha \in A\} \) is a collection of closed sets, then
\[
\bigcap_{\alpha \in A} C_\alpha
\]
is closed.

(c) If \( \{C_i : 1 \leq i \leq k\} \) is a finite collection of closed sets, then
\[
\bigcup_{i} C_i
\]
is closed.
Accumulation Points

An element \( x \) is an *accumulation point of the set \( A \) iff* for all \( \delta > 0 \), \( N(x,\delta) \) contains a point of \( A \) distinct from \( x \). Also called *limit points or cluster points*.

**Example:** Any interior point of a set is an accumulation point of the set.

For the set \( (3,5] \), both 3 and 5 are accumulation points.

So, an accumulation point need not belong to the set.

Any real number is an accumulation point of the set of rationals.

The set of accumulation points of a set \( A \) is called the *derived set* of \( A \) and denoted \( A' \).

**Example:** If \( A = (3,5] \) then \( A' = [3,5] \).

If \( A = \mathbb{Q} \) then \( A' = \mathbb{R} \).
Limit points

A real number \( x \) is a limit point of a subset \( A \) iff
\[
d(x, A\{x\}) = 0.
\]

Pf: Suppose that \( x \) is a limit point of \( A \), and pick \( \delta > 0 \).
Then \( N(x,\delta) \) is an open set containing a point \( y \) of \( A \) different from \( x \), and since \( y \in N(x,\delta) \), \( d(x,y) < \delta \). Since this holds for every \( \delta \), \( d(x,A\{x\}) = \inf \{d(x,y)\} = 0 \).

Now suppose that \( d(x,A\{x\}) = 0 \) and consider an open set \( O \) containing \( x \). Since \( O \) is open, there exists an \( N(x,\delta) \) entirely within \( O \). Since \( d(x,A\{x\}) < \delta \), there must be a point different from \( x \) but in \( A \) in \( N(x,\delta) \). So \( x \) is a limit point of \( A \).
Closed Sets

**Theorem:** A set is closed iff it contains all its accumulation points.

(i.e., $B$ is closed iff $B' \subseteq B$.)

**Pf:** Suppose that set $B$ is closed and $x$ is an accumulation point of $B$. BWOC assume that $x \notin B$. Then $x \in B^c$ which is an open set. Since $B^c$ is open, there is a $\delta$-neighborhood centered at $x$ contained entirely in $B^c$, and so, contains no point of $B$. $\rightarrow\leftarrow$ Thus, $x \in B$.

Now suppose that $B' \subseteq B$. Again, BWOC assume that $B^c$ is not open. Then, there is a point $y \in B^c$ which is not an interior point of $B^c$. Every $\delta$-neighborhood of $y$, must contain points of $B$ and so, $y$ is an accumulation point of $B$. $\rightarrow\leftarrow$ Thus, $B^c$ is open and so, $B$ is closed.
We can now return to the proof of $(b) \implies (c)$.

Let $x \in O$ which is the union of open intervals. Now $x$ must be in some open interval contained in $O$, say $(d,e)$. Now, if $d(x,\mathbb{R}\setminus O) = 0$, then $x$ would be a limit point of $\mathbb{R}\setminus O$ and every open set containing $x$ would contain a point of $\mathbb{R}\setminus O$. But $(d,e)$ is an open set containing no point of $\mathbb{R}\setminus O$. Therefore, for all $x \in O$, $d(x,\mathbb{R}\setminus O) > 0$. 
Cantor's Nested Interval Theorem

A sequence of sets \( \{S_n\} \) is **nested** if \( S_{i+1} \subset S_i \) for all \( i \).

**Theorem:** If \( \{[a_n,b_n]\} \) is a nested sequence of closed and bounded intervals, then \( \bigcap_n [a_n,b_n] \) is not empty. If, in addition, the width of the intervals \( |b_n-a_n| \) converges to 0 then \( \bigcap_n [a_n,b_n] \) has exactly one member.

**Pf:** Since the intervals are nested:
The left endpoints form an increasing sequence.
The right endpoints form a decreasing sequence.
Each left endpoint is less than or equal to each right endpoint.
Cantor's Nested Interval Theorem

**Theorem:** If \([a_n, b_n]\) is a nested sequence of closed and bounded intervals, then \(\bigcap_n [a_n, b_n]\) is not empty. If, in addition, the width of the intervals (\(|b_n - a_n|\)) converges to 0 then \(\bigcap_n [a_n, b_n]\) has exactly one member.

**Pf:** (cont): Let \(c\) denote the least upper bound of the left endpoints and \(d\) the inf of the right endpoints. The existence of \(c\) and \(d\) is guaranteed by the Least Upper Bound Property and the Greatest Lower Bound Property. Now \(c \leq b_n\) for all \(n\), so \(c \leq d\). Since \(a_n \leq c \leq d \leq b_n\), we have \([c, d] \subset [a_n, b_n]\) for each \(n\).

If the width of the intervals approach 0, then it follows that \(c = d\) and \(c\) is the single point of the intersection.
Covers

Let B be a set. A collection of sets $\mathcal{A}$ is a cover of B iff B is contained in the union of all the sets of $\mathcal{A}$.

A subcover of $\mathcal{A}$ for B is a subcollection of the sets of $\mathcal{A}$ which also cover B.

Example:
Let $B = (0, 1/2)$.
Let $\mathcal{A} = \{A_n \}$ where $A_n = [-1/n, 1/n)$
$\mathcal{A}$ is a cover of B.
$\{A_1, A_2\}$ is a subcover of $\mathcal{A}$ for B.
Compact Sets

A set $A$ is **compact** iff every cover of $A$ by open sets has a finite subcover.

**Examples:**
The empty set is compact.
Any finite set of points is a compact set.

The set $B = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ is a compact set.
Heine-Borel Theorem

Heine-Borel Theorem: Any closed interval \([a,b]\) of \(\mathbb{R}\) is compact.

\textit{Pf}: Let \(O\) be an open cover of \([a,b]\) and assume that \([a,b]\) is not compact. Divide the interval into two closed intervals \([a,\frac{1}{2}(a+b)]\) and \([\frac{1}{2}(a+b),b]\). Both of these can not have a finite subcover of \(O\). Pick one which does not have a finite subcover and call it \([a_1,b_1]\). Repeat this construction for the new interval to get \([a_2,b_2]\). This will ultimately give a nested sequence of closed intervals, none of which has a subcover of \(O\). The width of these intervals approaches 0. By Cantor's Nested Interval Theorem there is precisely one point, \(p\), common to all the intervals. The point \(p\) is in an open set of \(O\) and so, in an open \(N(p,\delta)\). The width of the \([a_n,b_n]\) will eventually become \(< \delta\) and so, contained in \(N(p,\delta)\) and thus a single open set of \(O\).
Heine-Borel Theorem

The Heine-Borel theorem can be generalized by replacing the closed interval by any closed and bounded set and in this more general form the result is if and only if.
Bolzano-Weierstrass Theorem

Bolzano-Weierstrass Theorem: Every bounded infinite subset of \( \mathbb{R} \) has an accumulation point in \( \mathbb{R} \).

\[ \text{Pf:} \quad \text{Suppose that } A \text{ is bounded and infinite, but has no accumulation points. Since } A' = \emptyset, A \text{ is closed. By the Heine-Borel theorem, } A \text{ is compact.} \]

\[ \text{Since } A \text{ has no accumulation points, for each } x \in A \text{ there exists a } \delta_x \text{-neighborhood which contains no element of } A \text{ other than } x. \text{ The set of all these neighborhoods is an infinite open cover of } A \text{ (with each open set containing only one point of } A). \text{ This open cover can have no finite subcover, contradicting the compactness of } A. \text{ Thus, } A \text{ must have an accumulation point.} \]
Completeness Equivalences

Completeness of \( \mathbb{IR} \)

Heine-Borel

Bounded Monotone

Bolzano-Weierstrass