The Cubic formula

Milan vs. Venice

(Cardano and Tartaglia)

1500's

Doubling the Cube

(The Delian Problem)

- To rid Athens of the plague, the cubic altar to Apollo must be doubled in size. 450 BC
- $x^3 = 2a^3$
- Solving for x would amount to being able to construct* the cube root of 2.
- (This is proved impossible by Wantzel in 1837.)

We move to 1200's in Italy

- John of Palerma proposes to Fibonacci that he solve $x^3 + 2x^2 + 10x = 20$.
- Fibonaacci only shows that there is no rational solution….

Fast forward to 1494

- Fra Luca Pacioli in *Summa …** asserts that a solution to the cubic equation was "as impossible as" squaring the circle.
- Around 1515 his colleague, Del Ferro, figures out how to solve cubics of the form

$$x^3 + px = q.$$

- He keeps the secret till his death bed…
- when he tells his student Fiore.

- ENTER TARTAGLIA (1500 - 1557)

- In 1535, he figures out how to solve equations of the form $x^3 + px^2 = q$ and announces so.

- Fiore challenges Tartaglia to a contest where each poses 30 problems for other to solve.
- Just before the contest, Tartaglia figures out how to solve cubic equations of the form $x^3 + px = q$. Now he knows how to solve two kinds of cubic equations.
- Fiore fails to solve all that he gets from Tartaglia.

ENTER CARDANO (1501 - 1576)

- ✓ Cardano begs Tartaglia for the secret methods.
- ✓ More later when Dennis and Geoff do their Round Table.

- Cardano was widely published - astrology, music, philosophy, and medicine.
- 131 works were published during his lifetime
- and 111 more were left in manuscript form.
- In mathematics, he wrote on a wide variety of subjects. Found among his papers was *Book on Games of Chance.* The work broke ground on theory of probability 50 years before Fermat and Pascal, but it wasn't published until 1663, the year after Pascal died.
- His greatest work was *Ars Magna* (The Great Art) published in 1545. It was the first Latin treatise devoted exclusively to algebra. (MACTUTOR)

Find two numbers whose product is 40 and whose sum is 10.

- Sound familiar?
- Let’s all make a table of pairs of whole numbers that sum to ten and check their products.
- Hmmm.

Try: $5+\sqrt{-15}$ and $5-\sqrt{-15}$

- *"Dismissing mental tortures, and multiplying 5 + √-15 by 5 - √-15, we obtain 25 - (-15). Therefore the product is 40. …. and thus far does arithmetical subtlety go, of which this, the extreme, is, as I have said, so subtle that it is useless."* (MACTUTOR)

- Even though Cardano - and the other Italian algebraists of the time - still would not consider equations with negative coefficients, he is willing to think about solutions that are complex numbers !

- “So progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless.”

Let's take a look at one of Cardano's innovations in his *Ars Magna.*

- ✓ We begin learning his idea by trying it out on a quadratic equation. The technique is now known as "depressing" as polynomial.

$$x^2 + 6x - 16 = 0$$

Because +6/-2 = -3, we will replace *x* by (*y*-3).

$$x^2 + 6x - 16 = 0$$

$$(y-3)^2 + 6(y-3) - 16 = 0$$

$$y^2 - 6y + 9 \quad + \quad 6y - 18 \quad - 16 \quad = 0$$

Notice how the -6*y* cancels with the +6*y*.

Collect like terms and notice that
the new equation has no linear term.

$$y^2 - 25 = 0$$

$$y^2 - 25 = 0$$

$$y = 5, -5$$

Since $x = y - 3$, $x = 2$ and $x = -8$ *are* solutions to the original equation.

Cardano is the first among his contemporaries to accept -8 as a solution. (Katz, p. 334)

This substitution technique is another example of a perfectly useful algebraic technique that is different from the ones that we have been taught.

Let us now apply this “depressing” technique to a cubic equation.

In the equation below, one would substitute $x = y - 2$.

$$x^3 + 6x^2 + 3x = 2$$

Since +6/-3 = -2, we use $y - 2$. (We skip details here.)

The resulting equation is $y^3 - 9y + 8 = 0$.

Notice that the squared term has been eliminated, so we consider that last equation a <u>depressed</u> cubic.

Cardano considers the equation: $x^3 = 15x + 4$.

He applies the cubic formula for this form of the equation and arrives at this "mess":

$$x = \sqrt[3]{(2+\sqrt{-121})} + \sqrt[3]{(2-\sqrt{-121})}$$

If you set your TI to complex mode, you can confirm that this complex formula is, in fact, equal to 4.

Enter Ferrari, Cardano's student

- He extends his teacher's techniques beginning with the step to put the fourth degree equation into depressed form.
- He then is able to find a solution by radicals for any fourth degree equation.
- Cardano includes Ferrari's result in *Ars Magna.*

Ferrari challenges Tartaglia …

Another good story…….

See Ferrari's listing on MACTUTOR.

Quintic (fifth degree) Polynomials

- 300 years go by as algebraists look for a formula or system by which they can solve fifth degree equations.
- Ruffini (1765 - 1822) proves that there can be no quintic formula, but has errors in his proof.

- Abel (1802 - 1829) studies the quintic -
- When he is 19, he proves that there can be no formula using roots for the general quintic polynomial.
- He self publishes his result, but …
- to save money he condenses his writing to a difficult-to-read six page pamphlet.

He sends it to Gauss and others …

- ✓ But no one takes much notice.
- ✓ Crelle, who is about to publish a journal and needs material, agrees to publish Abel’s proof and the word is out. (1827)

Abel-Ruffini Theorem:

"It is impossible to find a general formula for the roots of a polynomial equation of degree five or higher if the formula is allowed to use only arithmetic operations and the extraction of roots." (1824)

DEGREE		Can we always solve*?	Known Since
1	Linear equations	YES	1850BC
2	Quadratic equations	YES	1850BC
3	Cubic equations	YES	1545
4	Quartic equations	YES	1545
5	Quintic equations	IMPOSSIBLE	1824
higher		IMPOSSIBLE	1824

* Using only arithmetic operations and roots.

Thanks