On portfolio choice by maximising the outperformance probability

Anatolii A. Puhalskii
University of Colorado Denver and
Institute for Problems in Information Transmission

December 17, 2008

Abstract
We consider the problem of optimal portfolio selection for a multidimensional geometric Brownian motion model. We look for portfolios that maximise the probability of outperforming a stochastic benchmark. More specifically, we seek to maximise the decay rate of the shortfall probability and (or) to minimise the decay rate of the outperformance probability in the long run. A simple heuristic enables us to find an asymptotically optimal investment policy. The results provide interesting insights.

1 Introduction
Beating a benchmark or an index is a fairly common investment goal. Let $X_t$ denote the investor’s wealth at time $t$ and let $I_t$ denote the price of the benchmark. A portfolio’s performance is measured by the ratio $X_t/I_t$. Browne [1, 3] was one of the first to consider objective functions “related to the achievement of performance goals and shortfalls” in a continuous-time setting where both the assets comprising the portfolio and the benchmark evolve according to geometric Brownian motions. In particular, he considered the objective of maximising the probability that the wealth/benchmark ratio attains a given level before falling to a certain lower level. Infinite time horizon settings were addressed by Pham [6, 7] and Stutzer [8, 9, 10, 11] who were concerned with maximising the long term outperformance probability. Since the ratio $X_t/I_t$ typically grows or decays at an exponential rate, the relevant state variable is $L_t = (1/t) \ln (X_t/I_t)$ and the outperformance probability is defined as $P(L_t \geq q)$, where $q$ is a given threshold. Pham [6, 7] studied the problem of minimising the decay rate of the outperformance probability as $t \to \infty$ for a geometric Brownian motion model with one risky asset subject to the influence of an economic factor. (We say that the decay rate is $\kappa > 0$ if the probability in question decays as $\exp(-\kappa t)$.) The case of incomplete information and a deterministic benchmark was considered in Hata and Iida [5]. Stutzer [8, 9, 10, 11] advocated the criterion of maximising the decay rate of the shortfall probability $P(L^*_t < q)$ among all portfolios that outperform the benchmark with probability one. The specific examples he analysed were set in discrete time and assumed time-homogeneous controls.

The research reported in this paper was motivated by the desire to provide new tools and gain better insight into the settings investigated by Pham [6] and Stutzer [10]. Both these authors rely in their analyses on duality arguments. Stutzer [10] draws on Gärtner’s theorem which relates the decay rate to the asymptotics of the moment generating function. Pham [6] uses duality considerations in order to reformulate the problem as a risk sensitive control problem. He points out that the methods of risk sensitive control are not suitable for dealing with the shortfall probability. He also mentions the difficulties of extending the results on the decay rate of the outperformance probability to a multidimensional case.
Our primary goal is to provide a solution to the problem of maximising the decay rate of the shortfall probability. We study the model considered by Browne [3], i.e., an economic factor is not accounted for. On the other hand, we are concerned with a multidimensional setting and do not assume that the model is complete. As pointed out by Pham [6], this is a nonstandard control theory problem. Similarly to Pham [6] and Stutzer [10], we invoke large deviation theory techniques, however, our approach is more direct and does not use either duality considerations or the methods of stochastic control. We first give an intuitive heuristic argument that allows us to characterise the decay rate via a convex minimisation variational problem which admits an explicit solution. The minimiser furnishes an optimal investment policy. As is often the case with infinite horizon control problems, it is a constant proportion portfolio allocation policy (i.e., a time-homogeneous policy) which vindicates the choice of Stutzer [10] to restrict consideration to time-homogeneous policies. For a discussion of constant proportion portfolio allocation policies, see Browne [2]. A similar heuristic enables us to minimise the outperformance probability decay rate. The associated variational problem is in a certain sense complementary to the one for the shortfall probability optimisation. In particular, we find an asymptotically optimal policy. We recall that in Pham [6] only nearly optimal policies were obtained. The actual proofs that the heuristics yield optimal decay rates are quite different from the way the latter are arrived at and use standard tools from large deviation theory. We obtain a large deviation lower bound for the shortfall probability and a large deviation upper bound for the outperformance probability. We then show that the bounds are attained. The lower bound is established by using a change of probability measure and the proof of the upper bound is based on a version of Chernoff’s bound. However, an application of these tools to the problem at hand calls for certain ingenuity. In particular, the variational problems are instrumental in devising the proofs.

As a byproduct, we find out that the goals of optimising the outperformance and shortfall probabilities decay rates are not incompatible. For the outperformance criterion considered by Pham [6], the optimal decay rate is positive when $q$ is greater than a certain threshold and equals zero when $q$ is less than that threshold. Thus, for large values of $q$ one is concerned with portfolios that underperform and looks for the least unfavourable portfolio. On the other hand, for outperforming portfolios which correspond to smaller values of $q$ the criterion is not sensitive enough: if $\pi$ is any asymptotically outperforming portfolio, i.e., if $\lim_{t \to \infty} P(L_t \geq q) = 1$, then $\lim_{t \to \infty} (1/t) \ln P(L_t \geq q) = 0$. Therefore, for these portfolios Stutzer’s [10] shortfall criterion which minimises $\lim_{t \to \infty} (1/t) \ln P(L_t < q)$ should be used. This limit is negative for outperforming portfolios and equals zero for underperforming ones, which allows us to distinguish between outperforming portfolios and choose the one with the greatest shortfall probability decay rate. One would hope both to minimise the outperformance probability decay rate and to maximise the shortfall probability decay rate. Note that for a given $t$ maximising the outperformance probability $P(L_t \geq q)$ is equivalent to minimising the shortfall probability $P(L_t < q)$. We show that this can be achieved for the decay rates too and that under fairly general hypotheses there exists a constant proportion portfolio allocation policy that maximises $\limsup_{t \to \infty} (1/t) \ln P(L_t \geq q)$ and minimises $\liminf_{t \to \infty} (1/t) \ln P(L_t < q)$.

An outline of the remainder of this paper is as follows. In Section 2 the model is formulated, the heuristic argument is presented, and the main results are stated. Theorem 2.1 obtains the maximal decay rate of the shortfall probability, Theorem 2.2 is concerned with the outperformance probability, and Theorem 2.3 states the existence of an optimal control. The variational problems are studied in Section 3. We also identify the optimal control. In Sections 4 and 5 we prove Theorems 2.1 and 2.2, respectively. The arguments presented also prove Theorem 2.3, so we omit a separate proof of that result. Section 6 contains additional remarks which partly summarise our findings and partly discuss some specific settings.
2 A model description, heuristics, and the main results

We consider a portfolio consisting of \( n \) risky securities priced \( S^1_t, \ldots, S^n_t \) at time \( t \) and a safe security of price \( B_t \) at time \( t \). The risky securities are modelled by geometric Brownian motions, i.e., for \( i = 1, 2, \ldots, n \),

\[
\frac{dS^i_t}{S^i_t} = a^i dt + b^i \cdot dW_t
\]

(2.1)

where \( a^i \) is a real constant, \( b^i \) is a constant \( m \)-vector, \( W_t \) is an \( m \)-dimensional standard Wiener process, and \( S^i_0 > 0 \). We use \( \cdot \) to denote inner product. All vectors are considered as column vectors.

All processes are defined on a complete probability space \((\Omega, \mathcal{F}, P)\). It is assumed, furthermore, that the processes \( S^i_t = (S^i_t, t \in \mathbb{R}_+) \) are adapted to a filtration \( \mathcal{F} = (\mathcal{F}_t, t \in \mathbb{R}_+) \) and that \( W = (W_t, t \in \mathbb{R}_+) \) is a standard Wiener process relative to \( \mathcal{F} \).

The price of the safe security obeys the equation

\[
\frac{dB_t}{B_t} = r dt
\]

(2.2)

where \( r \) is a real constant and \( B_0 > 0 \). The investor holds \( l_i^t \) shares of risky security \( i \) and \( l_0^t \) shares of the safe security at time \( t \) so the total wealth \( X_t \) is given by

\[
X_t = \sum_{i=1}^{n} l_i^t S^i_t + l_0^t B_t.
\]

(2.3)

The portfolio

\[
\pi_t = (\pi^1_t, \ldots, \pi^n_t)
\]

specifies the proportions of the total wealth invested in the risky securities in that, for \( i = 1, 2, \ldots, n \),

\[
l_i^t S^i_t = \pi_i^t X_t.
\]

The processes \( \pi_t = (\pi^i_t, t \in \mathbb{R}_+) \) are assumed to be \( B \otimes \mathcal{F} \)-measurable, where \( B \) denotes the Borel \( \sigma \)-algebra on \( \mathbb{R}_+ \), \( \mathcal{F} \)-adapted, and such that \( \int_0^t \pi^i_s^2 \, ds < \infty \) a.s. We do not impose any other restrictions on the magnitudes of the \( \pi^i_t \) so that unlimited borrowing and shortselling are allowed.

According to (2.3), the amount of wealth invested in the safe security is \((1 - \sum_{i=1}^{n} \pi^i_t)X_t\). By the self-financing condition,

\[
\frac{dX_t}{X_t} = \sum_{i=1}^{n} \pi^i_t \frac{dS^i_t}{S^i_t} + \left(1 - \sum_{i=1}^{n} \pi^i_t\right) \frac{dB_t}{B_t}.
\]

(2.4)

The benchmark \( I = (I_t, t \in \mathbb{R}_+) \) follows the geometric Brownian motion

\[
\frac{dI_t}{I_t} = \alpha dt + \beta \cdot dW_t,
\]

(2.5)

where \( \alpha \) is a real constant, \( \beta \) is a constant \( m \)-vector, and \( I_0 > 0 \). The process \( I_t \) is also taken to be \( \mathcal{F} \)-adapted. We note that assuming that \( I_t \) is driven by the same Wiener process as the \( S^i_t \) is not a loss of generality.

Let \( a \) denote the \( n \)-vector with entries \( a^1, \ldots, a^n \), let \( b \) denote the \( n \times m \) matrix with rows \( b^1, \ldots, b^m \), \( c = bb^T \), and let \( \gamma = \|\beta\|^2 \), where \( ^T \) is used to denote the transpose of a matrix and \( \| \| \) is used to denote the Euclidean length of a vector. Let also \( \mathbf{1} \) denote the \( n \)-vector with unit entries.
The following derivation originates from Pham [6]. By Ito’s formula,
\[ d \ln X_t = \pi_t \cdot a \, dt + (b^T \pi_t) \cdot dW_t - \frac{1}{2} \pi_t \cdot c \pi_t \, dt + (1 - \pi_t \cdot 1) r \, dt, \]
\[ d \ln I_t = \alpha \, dt + \beta \cdot dW_t - \frac{1}{2} \gamma \, dt. \]

Accordingly, for \( Z_t = X_t / I_t, \)
\[ d \ln Z_t = d \ln X_t - d \ln I_t = (\pi_t \cdot a + (1 - \pi_t \cdot 1) r - \frac{1}{2} \pi_t \cdot c \pi_t - \alpha + \frac{1}{2} \gamma) \, dt + (b^T \pi_t - \beta) \cdot dW_t. \]

Therefore, for
\[ L_t^\pi = \frac{\ln Z_t}{t}, \]
we have
\[ L_t^\pi = \frac{\ln Z_0}{t} + \frac{1}{t} \int_0^t (\pi_s \cdot a + (1 - \pi_s \cdot 1) r - \frac{1}{2} \pi_s \cdot c \pi_s - \alpha + \frac{1}{2} \gamma) \, ds + \frac{1}{t} \int_0^t (b^T \pi_s - \beta) \cdot dW_s. \quad (2.6) \]

Suppose that given \( q \in \mathbb{R}, \) we seek to maximise the decay rate of \( P(L_t^\pi < q) \) as \( t \to \infty \) over all admissible portfolios. More precisely, we look for \( \lim \inf_{t \to \infty} (1/t) \inf_{\pi} \ln P(L_t^\pi < q). \) The following heuristic argument provides useful insight.

Denote, for \( u \in \mathbb{R}^n, \)
\[ M(u) = u \cdot (a - r 1) - \frac{1}{2} u \cdot c u + r - \alpha + \frac{1}{2} \gamma, \quad (2.7) \]
\[ N(u) = b^T u - \beta. \quad (2.8) \]

By a change of variables, we can bring (2.6) to the form
\[ L_t^\pi = \frac{\ln Z_0}{t} + \int_0^1 M(\pi^t_s) \, ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi^t_s) \cdot dW^t_s, \quad (2.9) \]
where \( W^t_s = W_{st}/\sqrt{t} \) and \( \pi^t_s = \pi_{st}. \) Note that \( W^t = (W^t_s, s \in [0,1]) \) is a Wiener process relative to \( \mathbb{F}^t = (\mathcal{F}_{st}, s \in [0,1]) \) and \( \pi^t = (\pi^t_s, s \in [0,1]) \) is \( \mathbb{F}^t \)-adapted.

Suppose that \( Z_0 = 1. \) Let \( \mathcal{A} \) be a set of \( \mathbb{R} \)-valued functions on \( [0,1] \) such that \( \sup_{\pi^t \in \mathcal{A}} L_t^\pi \) is a well-defined random variable for all \( t \in \mathbb{R}_+. \) Note that \( \pi^t \) is taken to be a deterministic function when this supremum is considered. Clearly,
\[ \inf_{\pi: \pi^t(\omega) \in \mathcal{A}} P(L_t^\pi < q) \geq P\left( \sup_{\pi^t \in \mathcal{A}} L_t^\pi < q \right), \]
so,
\[ \lim \inf_{t \to \infty} \frac{1}{t} \inf_{\pi: \pi^t(\omega) \in \mathcal{A}} \ln P(L_t^\pi < q) \geq \lim \inf_{t \to \infty} \frac{1}{t} \ln P\left( \sup_{\pi^t \in \mathcal{A}} L_t^\pi < q \right). \]

Recall that the net \( W^t/\sqrt{t} \) obeys the large deviation principle in \( C([0,1], \mathbb{R}^m) \) for rate \( t \) as \( t \to \infty \) with action functional \( I^W \) given by \( (1/2) \int_0^1 \|\dot{w}_s\|^2 \, ds \) if \( w = (w_t, t \in [0,1]) \) is an absolutely
continuous \( \mathbb{R}^m \)-valued function with \( w_0 = 0 \) and being equal to infinity otherwise, see, e.g., Freidlin and Wentzell [4]. Here \( \dot{w}_s \) denotes the derivative. If we suppose that
\[
\sup_{\pi^t \in A} L^\pi_t = \sup_{\pi^t \in A} \left( \int_0^1 M(\pi^t_s) \, ds + \frac{1}{\sqrt{t}} \int_0^1 N(\pi^t_s) \cdot dW^s_t \right)
\]
is a suitably continuous function of \( W^t / \sqrt{t} \), then by the continuous mapping principle,
\[
\liminf_{t \to \infty} \frac{1}{t} \ln P(\sup_{\pi^t \in A} L^\pi_t < q) \geq - \inf \left( I^W(w) + \sup_{\pi^t \in A} \int_0^1 \left( M(\pi_s) + N(\pi_s) \cdot \dot{w}_s \right) ds < q \right) .
\] (2.10)

This prompts introducing
\[
J_q = \inf \left( \frac{1}{2} \int_0^1 \|f_s\|^2 ds; \sup_{u \in \mathbb{R}^n} \left( M(u) + N(u) \cdot f_s \right) ds \leq q \right) ,
\] (2.11)
where \( (f_s) \in L_2([0, 1], \mathbb{R}^m) \) and infima over empty sets are assumed to be equal to \( \infty \). This is a convex minimisation problem. If the set of constraints in (2.11) is nonempty, then (2.11) has a unique minimiser. We note that \( J_q \) is a nonincreasing rightcontinuous function of \( q \) and denote by \( J_{q-} \) its left-hand limit at \( q \). We show below in Lemma 3.1 that the only way for \( J_q \) not to be continuous is for it “to jump to infinity”, i.e., for there being a \( q \) with \( J_{q-} = \infty \) and \( J_q < \infty \).

Since the righthand side of (2.10) is bounded from below by \(-J_{q-}\), on extending \( A \) to the set of all controls, we arrive at the following conjecture
\[
\liminf_{t \to \infty} \frac{1}{t} \ln \inf_{\pi^t \in \mathcal{U}} P(L^\pi_t < q) \geq - J_{q-}.
\]
The next theorem bears out the conjecture and shows that this bound is tight. Let \( \mathcal{U} \) denote the class of portfolios such that for arbitrary \( \epsilon > 0 \)
\[
\limsup_{t \to \infty} \sup_{\pi^t \in \mathcal{U}} P\left( \frac{1}{t^2} \int_0^t \|\pi_s\|^2 ds > \epsilon \right)^{1/t} = 0 .
\] (2.12)
The following hypothesis is assumed in all subsequent results:
\[
\lim_{t \to \infty} P\left( \frac{|\ln Z_0|}{t} > \epsilon \right)^{1/t} = 0 \quad \text{for arbitrary } \epsilon > 0 .
\] (2.13)

**Theorem 2.1.** Under the introduced hypotheses,
\[
\liminf_{t \to \infty} \frac{1}{t} \ln \inf_{\pi^t \in \mathcal{U}} P(L^\pi_t < q) \geq - J_{q-}
\]
and
\[
\limsup_{t \to \infty} \frac{1}{t} \ln \inf_{\pi^t \in \mathcal{U}} P(L^\pi_t \leq q) \leq - J_q .
\]

If \( J_{q-} < \infty \), then \( J_q = J_{q-} \), so
\[
\lim_{t \to \infty} \frac{1}{t} \ln \inf_{\pi^t \in \mathcal{U}} P(L^\pi_t < q) = \lim_{t \to \infty} \frac{1}{t} \ln \inf_{\pi^t \in \mathcal{U}} P(L^\pi_t \leq q) = - J_q .
\]
A similar approach applies to minimising the outperformance probability decay rate. If we note that

\[ \sup_{\pi: \pi'(\omega) \in A} P(L_t^\pi \geq q) \leq P(\sup_{\pi' \in A} L_t^\pi \geq q) \]

and suppose that

\[ \limsup_{t \to \infty} \frac{1}{t} \ln P(\sup_{\pi' \in A} L_t^\pi \geq q) \leq -\inf \left( \int W(w) \; sup_{\pi \in A} \left( M(\pi_s) + N(\pi_s) : w_s \right) ds \geq q \right) , \]

we can conjecture that

\[ \limsup_{t \to \infty} \frac{1}{t} \sup_{\pi} \ln P(L_t^\pi \geq q) \leq -J'_q , \]

where

\[ J'_q = \inf \left( \frac{1}{2} \int_0^1 \| f_s \|^2 ds ; \sup_{u \in \mathbb{R}^n} \left( M(u) + N(u) : f_s \right) ds \geq q \right) \]

(2.14)

with \((f_s)\in L_2([0,1],\mathbb{R}^m)\). We note that unlike (2.11) the constraint set is not convex. The function \(J'_q\) is nondecreasing. It follows by Lemma 3.1 below that \(J'_q\) is leftcontinuous and is continuous unless there is \(q\) with \(J'_q < \infty\) and \(J'_{q^+} = \infty\), where \(J'_{q^+}\) denotes the righthand limit.

The next theorem confirms the conjecture.

**Theorem 2.2.** Under the introduced hypotheses,

\[ \limsup_{t \to \infty} \frac{1}{t} \ln \sup_{\pi \in \mathcal{U}} P(L_t^\pi \geq q) \leq -J'_q \]

and

\[ \liminf_{t \to \infty} \frac{1}{t} \ln \sup_{\pi \in \mathcal{U}} P(L_t^\pi > q) \geq -J'_{q^+} . \]

If \(J'_{q^+} < \infty\), then \(J'_q = J'_{q^+}\), so

\[ \lim_{t \to \infty} \frac{1}{t} \ln \sup_{\pi \in \mathcal{U}} P(L_t^\pi > q) = \lim_{t \to \infty} \frac{1}{t} \ln \sup_{\pi \in \mathcal{U}} P(L_t^\pi \geq q) = -J'_q . \]

On comparing (2.11) and (2.14), we can see that \(J_q\) and \(J'_q\) are complementary in the sense that \(\min(J_q, J'_{q^+}) = \min(J_q^-, J'_{q^+}) = 0\) and, as we shall see, \(\max(J_q, J'_{q^+}) = \max(J_q^-, J'_{q^+}) > 0\). Therefore, if \(J_q = J_q^-\) and \(J'_q = J'_{q^+}\), then either \(\lim_{t \to \infty} (1/t) \ln \inf_{\pi \in \mathcal{U}} P(L_t^\pi \leq q) = 0\) and \(\lim_{t \to \infty} (1/t) \ln \sup_{\pi \in \mathcal{U}} P(L_t^\pi \geq q) < 0\) or \(\lim_{t \to \infty} (1/t) \ln \inf_{\pi \in \mathcal{U}} P(L_t^\pi \leq q) < 0\) and \(\lim_{t \to \infty} (1/t) \ln \sup_{\pi \in \mathcal{U}} P(L_t^\pi \geq q) = 0\).

The next theorem shows that the bounds in Theorems 2.1 and 2.2 are attained by a constant proportion portfolio.

**Theorem 2.3.** Given \(q \in \mathbb{R}_+\), there exists vector \(\hat{u} \in \mathbb{R}^n\) such that the portfolio \(\hat{\pi}\) with \(\hat{\pi}_t = \hat{u}\) enjoys the following properties:

If \(J_q^- < \infty\), then

\[ \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^{\hat{\pi}} < q) = \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^{\hat{\pi}} \leq q) = -J_q ; \]

If \(J'_{q^+} < \infty\), then

\[ \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^{\hat{\pi}} > q) = \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^{\hat{\pi}} \geq q) = -J'_q . \]
We note that if $J_q - J > 0$, then $\lim_{t \to \infty} P(L_t^\pi < q) = 0$, so $\lim_{t \to \infty} P(L_t^\pi \geq q) = 1$ and $\hat{\pi}$ is an almost surely outperforming portfolio, which corresponds to the framework considered by Stutzer [10]. The definition of the vector $\hat{u}$ is given in Section 3. Loosely speaking, it is the optimal $u$ in (2.11) and (2.14). Besides, once we know that an optimal portfolio is given by a constant vector, it is an easy matter to evaluate the outperformance and shortfall probabilities and identify $\hat{u}$. In Section 3 also we give a complete classification of cases where $J_q - J$, $J_q$, $J'_q$, and $J'_q +$ are finite. In particular, Lemma 3.1 states sufficient conditions in terms of the coefficients of equations (2.1), (2.2), and (2.5).

Let us also note that Pham [6] uses a slightly different optimality criterion which is $\sup_{\pi \in U} \lim_{t \to \infty} t^{-1} \ln P(L_t^\pi \geq q)$. One could also consider an analogous criterion for the shortfall probability as alluded to in Stutzer [10]. It is not difficult to see, however, that the concept of optimality in this paper is consistent with the one in Pham [6] and Stutzer [10] and that the control of Theorem 2.3 is also optimal in the other framework. Indeed, since

$$
\inf_{\pi \in U} \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^\pi < q) \geq \lim_{t \to \infty} \inf_{\pi \in U} \frac{1}{t} \ln P(L_t^\pi < q)
$$

and

$$
\sup_{\pi \in U} \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^\pi \geq q) \leq \lim_{t \to \infty} \sup_{\pi \in U} \frac{1}{t} \ln P(L_t^\pi \geq q),
$$

we obtain, as a byproduct of Theorems 2.1, 2.2, and 2.3 that if $J_q < \infty$, then

$$
\inf_{\pi \in U} \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^\pi < q) = \inf_{\pi \in U} \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^\pi \leq q) = \inf_{\pi \in U} \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^\pi \leq q) = -J_q
$$

and if $J'_q < \infty$, then

$$
\sup_{\pi \in U} \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^\pi \geq q) = \sup_{\pi \in U} \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^\pi > q) = \sup_{\pi \in U} \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^\pi > q) = -J'_q.
$$

3 Solving the variational problems

In this section we solve problems (2.11) and (2.14). We also look for a vector $\hat{u}$ such that

$$
J_q = \inf \left( \frac{1}{2} \|v\|^2; \; M(\hat{u}) + N(\hat{u}) \cdot v \leq q \right)
$$

and (or)

$$
J'_q = \inf \left( \frac{1}{2} \|v\|^2; \; M(\hat{u}) + N(\hat{u}) \cdot v \geq q \right)
$$

which is used for defining the optimal control $\hat{\pi}$.

The following observation will enable us to simplify certain arguments. By Jensen’s inequality, (2.11) can be equivalently written as a finite-dimensional optimisation problem:

$$
J_q = \inf \left( \frac{1}{2} \|v\|^2; \; \sup_{u \in \mathbb{R}^n} (M(u) + N(u) \cdot v) \leq q \right).
$$

7
Besides, we introduce the notation $y^+ = \max(y, 0)$ where $y \in \mathbb{R}$.

Let us first consider the case where $a - r \mathbf{1}$ is in the range of $b$ which coincides with the range of $c$. Then, by (2.7) and (2.8),

$$\sup_{u \in \mathbb{R}^n} (M(u) + N(u) \cdot v) = \frac{1}{2} v \cdot b^T c^\dagger b v - p \cdot v + w,$$

(3.4)

where $c^\dagger$ denotes the pseudo inverse of $c$,

$$p = \beta - b^T c^\dagger (a - r \mathbf{1}),$$

(3.5)

$$w = \frac{1}{2} (a - r \mathbf{1}) \cdot c^\dagger (a - r \mathbf{1}) + (r - \alpha + \frac{1}{2} \gamma).$$

(3.6)

The supremum is attained at

$$u(v) = c^\dagger (a - r \mathbf{1} + bv).$$

(3.7)

Note that by (2.8) and (3.5),

$$N(u(v)) = b^T c^\dagger bv - p,$$

(3.8)

and by (2.7) and (3.6),

$$M(c^\dagger (a - r \mathbf{1})) = w.$$  

(3.9)

It is convenient to introduce orthogonal decompositions of vectors into the sums of their projections onto the range of $b^T$ and onto the nullspace of $b$: for $x \in \mathbb{R}^n$, we will denote as $x^{(1)}$ the orthogonal projection of $x$ onto the range of $b^T$ and as $x^{(2)}$, the orthogonal projection of $x$ onto the nullspace of $b$. They are given by the respective equalities $x^{(1)} = b^T c^\dagger b x$ and $x^{(2)} = x - b^T c^\dagger b x$.

We have by (3.3), (3.4), and (2.14), on noting that $p^{(2)} = \beta^{(2)}$, that

$$J_q = \frac{1}{2} \inf \left( \| v^{(1)} \|^2 + \| v^{(2)} \|^2; \frac{1}{2} \| v^{(1)} \|^2 - p^{(1)} \cdot v^{(1)} - \beta^{(2)} \cdot v^{(2)} + w \leq q \right)$$

(3.10)

and

$$J'_q = \frac{1}{2} \inf \left( \int_0^1 \| f_s^{(1)} \|^2 ds + \int_0^1 \| f_s^{(2)} \|^2 ds; \frac{1}{2} \int_0^1 \| f_s^{(1)} \|^2 ds - \int_0^1 p^{(1)} \cdot f_s^{(1)} ds - \int_0^1 \beta^{(2)} \cdot f_s^{(2)} ds + w \geq q \right).$$

(3.11)

We can thus optimise successively in the two orthogonal subspaces. We will mark with $\hat{\cdot}$ vectors that deliver minima unless mentioned otherwise. We work with $J_q$ first. Suppose, in addition, that $\beta$ is not in the range of $b^T$, so $\| \beta^{(2)} \| > 0$. We have with the use of the Cauchy-Schwartz inequality that

$$\inf \left( \| v^{(2)} \|^2; \frac{1}{2} \| v^{(1)} \|^2 - p^{(1)} \cdot v^{(1)} - \beta^{(2)} \cdot v^{(2)} + w \leq q \right) = \frac{1}{\| \beta^{(2)} \|^2} \left( \frac{1}{2} \| v^{(1)} \|^2 - p^{(1)} \cdot v^{(1)} + w - q \right)^+.$$ 

(3.12)

and that the infimum is attained at

$$v^{(2)} = \frac{\beta^{(2)}}{\| \beta^{(2)} \|^2} \left( \frac{1}{2} \| v^{(1)} \|^2 - p^{(1)} \cdot v^{(1)} + w - q \right)^+. $$

(3.12)

By (3.10) and a simple algebra,

$$J_q = \frac{1}{2} \inf \left( \| v^{(1)} \|^2 + \frac{1}{\| \beta^{(2)} \|^2} \left( \frac{1}{2} \| v^{(1)} \|^2 - d \right)^+ \right),$$

(3.13)
that the infimum equals infinity if $d < \|v\|^2/2 - d \geq 0$, it is convenient to introduce $\hat{v} = v^{(1)} - p^{(1)}$. The expression in the infimum on the right of (3.13) takes the form

$$
\|\hat{v}\|^2 + 2\hat{v} \cdot p^{(1)} + \|p^{(1)}\|^2 + \frac{1}{\|\beta(2)\|^2} \left( \frac{1}{2} \|\hat{v}\|^2 - d \right)^2.
$$

By the Cauchy-Schwartz inequality it is bounded from below by

$$
\|\hat{v}\|^2 + 2\|\hat{v}\|\|p^{(1)}\| + \|p^{(1)}\|^2 + \frac{1}{\|\beta(2)\|^2} \left( \frac{1}{2} \|\hat{v}\|^2 - d \right)^2.
$$

The infimum of the righthand side of (3.13) over the region $\|v^{(1)} - p^{(1)}\|^2/2 - d \geq 0$, it is positive for all $z > \hat{v} = 0$. If $\|p^{(1)}\| = 0$, then the lefthand side of (3.17) is positive at $z = 0$. It follows that an optimal $\hat{v}$ is of the form $\hat{v} = -(z/\|p^{(1)}\|)p^{(1)}$. If $\|p^{(1)}\| = 0$, then the lefthand of (3.17) is positive at $z = 0$. It follows that it is positive for all $z > 0$, hence, (3.16) is positive for all $z > 0$, which implies that the minimum of (3.15) is attained at $\hat{v} = 0$. It also satisfies the condition that $\|\hat{v}\|^2 \geq 2d$. We let $\hat{v} = 0$ if (3.17) has no solution.

Consequently, it is optimal to take

$$
\hat{v}^{(1)} = \begin{cases} 
(1 - \frac{\hat{v}}{\|p^{(1)}\|})p^{(1)}, & \text{if } p^{(1)} \neq 0, \\
0, & \text{if } p^{(1)} = 0,
\end{cases}
$$

and by (3.12)

$$
\hat{v}^{(2)} = \frac{\beta(2)}{\|\beta(2)\|^2} \left( \frac{\hat{v}^2}{2} - d \right).
$$

The infimum of the righthand side of (3.13) over the region $\|v^{(1)} - p^{(1)}\|^2/2 - d \geq 0$ equals $\|\hat{v}^{(1)}\|^2/2 + \|\hat{v}^{(2)}\|^2/2$. It depends continuously on $q$. We also note that $\hat{v}^{(2)} \neq 0$.

If we consider the righthand side of (3.13) on the region $\|v^{(1)} - p^{(1)}\|^2/2 - d \leq 0$, we can see that the infimum equals infinity if $d < 0$. If $d \geq 0$, then this region is a ball centered at the
endpoint of \( p^{(1)} \) which does not contain the origin. Note that \( p^{(1)} \neq 0 \) under the assumptions that \( q < w \) and \( d \geq 0 \). Therefore, the vector \( v^{(1)} \) of minimum norm from this ball belongs to the boundary \( \|v^{(1)} - p^{(1)}\|^2/2 - d = 0 \). (One could also invoke the Cauchy-Schwartz inequality.) But the boundary has been accounted for in the preceding optimisation. We conclude that \( J_q = \|\hat{v}\|^2/2 \), where \( \hat{v} = \hat{v}^{(1)} + \hat{v}^{(2)} \). Also, \( J_{w-} = 0 \). In addition, since \( \hat{v}^{(2)} \neq 0 \), the infimum in (3.10) is attained at the boundary of the set of constraints, so by (3.4) \( \sup_{u \in \mathbb{R}^n} (M(u) + N(u) \cdot \hat{v}) = q \), and by (3.7),

\[
M(\hat{u}) + N(\hat{u}) \cdot \hat{v} = q, \quad (3.20)
\]

where

\[
\hat{u} = c^\dagger (a - r 1 + b \hat{v}). \quad (3.21)
\]

If \( q \geq w \), then \( J_q = 0 \).

Suppose \( \beta^{(2)} = 0 \). Then \( p^{(1)} = p \) and by (3.10)

\[
J_q = \frac{1}{2} \inf \left( \|v^{(1)}\|^2; \frac{1}{2} \|v^{(1)}\|^2 - p \cdot v^{(1)} + w \leq q \right),
\]

where \( v^{(1)} \) belongs to the range of \( b^T \). If \( q < w \) and, furthermore, \( q < w - \|p\|^2/2 \), then \( J_q = \infty \) as the constraint set is empty. In particular, \( J_{(w-\|p\|^2/2)-} = \infty \). If \( w - \|p\|^2/2 \leq q < w \), then the “shortest” \( v^{(1)} \) from the set of constraints must be the multiple of \( p \) that belongs to the sphere \( \|v^{(1)} - p\|^2/2 = d \) (note that \( p \neq 0 \) in this case). Therefore,

\[
\hat{v} = \hat{v}^{(1)} = \lambda p, \quad (3.22)
\]

where \( \lambda \in (0, 1] \) and

\[
(\lambda - 1)^2 = \frac{2d}{\|p\|^2}. \quad (3.23)
\]

Accordingly, \( J_q = \lambda^2\|p\|^2/2 \), so \( J_{w-} = 0 \). In addition, since \( \hat{v} \) belongs to the boundary of the set of constraints, we have (3.20) with \( \hat{u} \) from (3.21) for \( w - \|p\|^2/2 \leq q < w \). This case can also be described by (3.18) and (3.19) with \( \hat{z} = \sqrt{2d} \). If \( q \geq w \), then \( J_q = 0 \).

We now analyse \( J'_q \). We follow a similar strategy. Suppose that \( \|\beta^{(2)}\| > 0 \). Then the Cauchy-Schwartz inequality shows that

\[
\inf \left( \frac{1}{0} \int ||f_s^{(2)}||^2 \, ds; \frac{1}{2} \int ||f_s^{(1)}||^2 \, ds - \int \frac{1}{0} p^{(1)} \cdot f_s^{(1)} \, ds - \int \frac{1}{0} \beta^{(2)} \cdot f_s^{(2)} \, ds + w \geq q \right)
\]

\[
= \frac{1}{\|\beta^{(2)}\|^2} \left( (q - w - \frac{1}{2} \int \frac{1}{0} ||f_s^{(1)}||^2 \, ds + \int \frac{1}{0} p^{(1)} \cdot f_s^{(1)} \, ds)^+ \right)^2
\]

and that the infimum is attained at

\[
f_s^{(2)} = -\frac{\beta^{(2)}}{\|\beta^{(2)}\|^2} \left( q - w - \frac{1}{2} \int \frac{1}{0} ||f_s^{(1)}||^2 \, ds + \int \frac{1}{0} p^{(1)} \cdot f_s^{(1)} \, ds \right)^+. \quad (3.24)
\]

By (3.11),

\[
J'_q = \frac{1}{2} \inf \left( \frac{1}{0} \int ||f_s^{(1)}||^2 \, ds + \frac{1}{\|\beta^{(2)}\|^2} \left( (d - \frac{1}{2} \int \frac{1}{0} ||f_s^{(1)}||^2 \, ds)^+ \right)^2 \right), \quad (3.25)
\]
where the infimum is taken over \((f_s^{(1)}) \in L_2([0,1], \mathbb{R}^m)\) with \(f_s^{(1)}\) being in the range of \(b^T\).

If \(q \leq w\), then the infimum is attained at \(f_s^{(1)} = 0\), so \(J'_q = 0\). Suppose that \(q > w\). Minimisation over the region \(d - \int_0^1 \|f_s^{(1)} - p^{(1)}\| ds/2 \geq 0\) is carried out in a similar fashion as above. Let \(\tilde{f}_s = f_s^{(1)} - p^{(1)}\). Then we need to minimise

\[
\int_0^1 \|\tilde{f}_s\|^2 ds + 2 \int_0^1 p^{(1)} \cdot \tilde{f}_s ds + \|p^{(1)}\|^2 + \frac{1}{\|\beta^{(2)}\|^2} \left( d - \frac{1}{2} \int_0^1 \|f_s^{(1)}\|^2 ds \right)^2
\]

(3.26)

over the region \(\int_0^1 \|\tilde{f}_s\|^2 ds/2 \leq d\). As before, we bound it from below by

\[
\int_0^1 \|\tilde{f}_s\|^2 ds - 2\|p^{(1)}\| \sqrt{\int_0^1 \|\tilde{f}_s\|^2 ds + \|p^{(1)}\|^2 + \frac{1}{\|\beta^{(2)}\|^2} \left( d - \frac{1}{2} \int_0^1 \|f_s^{(1)}\|^2 ds \right)^2},
\]

(3.27)

find the minimum of the latter expression and identify \(\tilde{f}_s\) for which (3.26) equals (3.27). The minimum for (3.27) equals the minimum for (3.15). As we have seen, if \(\|p^{(1)}\| > 0\), the minimum of (3.15) is attained at \(\sqrt{\int_0^1 \|\tilde{f}_s\|^2 ds} = \hat{z}\). Since \(q > w\), \(\|p^{(1)}\| < \sqrt{2d}\). Consequently, the lefthand side of (3.17) is negative at \(z = \|p^{(1)}\|\) and is positive at \(z = \sqrt{2d}\). Therefore, the solution of (3.17) belongs to the interval \((\|p^{(1)}\|, \sqrt{2d})\), so it satisfies the requirement that \(\hat{z}^2/2 \leq d\). (Note that by (3.14), \(d > 0\) when \(q > w\).) We conclude that an optimal \(\hat{f}_s\) is of the form \(\hat{f}_s = -(\hat{z}/\|p^{(1)}\|)p^{(1)}\). If \(\|p^{(1)}\| = 0\) and \(q - w \leq \|\beta^{(2)}\|^2\), then the infimum of (3.26) is attained at \(\hat{f}_s = 0\) by the same argument as before. It is in the admissible region. However, if \(q - w > \|\beta^{(2)}\|^2\), then equation (3.17) has the unique positive solution \(\hat{z} = \sqrt{2(q - w - \|\beta^{(2)}\|^2)}\), which delivers infimum to (3.27). Then we can take as an optimal \(\hat{f}_s\) any vector from the range of \(b^T\) which is of magnitude \(\hat{z}\) provided \(b \neq 0\). We conclude that \(f_s^{(1)} = \hat{v}^{(1)}\), where \(\hat{v}^{(1)}\) is defined by (3.18) if either \(\|p^{(1)}\| > 0\) or \(\|p^{(1)}\| = 0\) and \(q \leq w + \|\beta^{(2)}\|^2\), and \(\hat{v}^{(1)}\) is an arbitrary vector of magnitude \(\hat{z}\) from the range of \(b^T\) if \(\|p^{(1)}\| = 0\), \(q > w + \|\beta^{(2)}\|^2\), and \(b \neq 0\). Substitution into (3.24) yields \(\tilde{f}_s^{(2)} = \hat{v}^{(2)}\), where \(\hat{v}^{(2)}\) is given by (3.19) and \(\hat{z} = 0\) if (3.17) has no solution. If \(\|p^{(1)}\| = 0\), \(q > w + \|\beta^{(2)}\|^2\), and \(b = 0\), then \(\hat{v}^{(1)} = 0\) and \(\hat{v}^{(2)} = (\beta^{(2)}/\|\beta^{(2)}\|^2)(w - q)\).

The set of functions \((f_s^{(1)})\) such that \(d - \int_0^1 \|f_s^{(1)} - p^{(1)}\|^2 ds/2 \leq 0\) is the exterior of a ball in \(L_2([0,1], \mathbb{R}^{m'})\), where \(m'\) is the dimension of the range of \(b^T\), centered at the endpoint of \(p^{(1)}\) and containing the origin. By geometric properties of Hilbert spaces, the infimum of \(\int_0^1 \|f_s^{(1)}\|^2 ds\) over this region is attained at the boundary \(d - \int_0^1 \|f_s^{(1)} - p^{(1)}\|^2 ds/2 = 0\). (A rigorous argument would use the Cauchy-Schwartz inequality.) The value of the objective function on this boundary cannot be less than the minimum over the region \(d - \int_0^1 \|f_s^{(1)} - p^{(1)}\|^2 ds/2 \geq 0\), which we have already found. By (3.11), \(J'_q = \|\hat{v}\|^2/2\), where \(\hat{v} = \hat{v}^{(1)} + \hat{v}^{(2)}\), also \(J'_w = 0\). Besides, we have (3.20) with \(\hat{v}\) from (3.21).

Suppose \(\beta^{(2)} = 0\). By (3.11),

\[
J'_q = \frac{1}{2} \inf \left( \int_0^1 \|f_s^{(1)}\|^2 ds; \frac{1}{2} \int_0^1 \|f_s^{(1)}\|^2 ds - \int_0^1 p \cdot f_s^{(1)} ds + w \geq q \right).
\]

If \(q \leq w\), then \(J'_q = 0\). Suppose \(q > w\). The set of constraints is of the form \(\int_0^1 \|f_s^{(1)} - p\|^2 ds/2 \geq d\), so it is the exterior of a ball in \(L_2([0,1], \mathbb{R}^{m'})\), which contains the origin. Therefore, if \(p \neq 0\), then
the “shortest” element of $L_2([0,1],\mathbb{R}^m')$ from the set of constraints is given by the intersection of the line through the endpoint of $p$ and the origin with the boundary of the ball, so it is of the form $f^1_s = \lambda p$, where $\lambda < 0$. It is found from the condition $(\lambda - 1)^2 \|p\|^2 = 2d$. Note that $p$ is in the range of $b^T$, hence, $(f^1_s)$ is too. Thus, $J'_q = \lambda^2 \|p\|^2/2$, in particular, $J'_{w+} = 0$. We can also see that $f^1_s = \hat{\nu}(1)$, where $\hat{\nu}(1)$ is given by (3.18) with $\hat{z} = \sqrt{2d}$. If $p = 0$ and $b \neq 0$, we take as a minimiser $f^1_s = \hat{\nu}$, where $\|\hat{\nu}\|^2 = 2(q - w)$ and $\hat{\nu}$ is in the range of $b^T$. (One can interpret this vector as a limit of (3.18) and (3.19) with $\hat{z} = \sqrt{2d}$.) Then $J'_q = q - w$ and $J'_{w+} = 0$. If $p = 0$ and $b = 0$, then the range of $b^T$ contains only the zero element. In this case, the constraint set is empty. We obtain that $J'_q = \infty$ and $J'_{w+} = \infty$.

Let us address the issue of validity of (3.1) and (3.2). Suppose that $\beta$ is not in the range of $b^T$. The vectors $\hat{\nu}(1) - p$ and $\hat{\nu} = \hat{\nu}(1) + \hat{\nu}(2)$ as defined by (3.18) and (3.19) are seen to be colinear. More specifically, $\hat{\nu} = (1 - \|p(1)\|/\hat{z})(\hat{\nu}(1) - p)$ if $p(1) \neq 0$ and $\hat{\nu} = -(w - q)/\|\beta(2)\|^2(\hat{\nu}(1) - p)$ if $p(1) = 0$. Consider the case where $q < w$. Since $\hat{\nu}(1) - p = N(\hat{u})$ by (3.8) and $\hat{z} < \|p(1)\|$ when $p(1) \neq 0$, we obtain that $\hat{\nu} = -\hat{\lambda} N(\hat{u})$ with $\hat{\lambda} > 0$ and that $\hat{\nu} \cdot N(\hat{u}) < 0$. By (3.20), $M(\hat{u}) > q$. Consequently, a vector $\hat{\nu}$ of minimum norm such that $M(\hat{u}) + N(\hat{u}) \cdot \hat{\nu} \leq q$ must be colinear with $N(\hat{u})$, i.e., $\hat{\nu} = -\hat{\lambda} N(\hat{u})$ for some $\hat{\lambda} > 0$. Since $M(\hat{u}) + N(\hat{u}) \cdot \hat{\nu} = q$ and $M(\hat{u}) + N(\hat{u}) \cdot \hat{\nu} = q = 0 = N(\hat{u}) \cdot (\hat{\nu} - \hat{\nu}(1)) = (\hat{\lambda} - \hat{\lambda})N(\hat{u})^2$, which implies that $\hat{\lambda} = \hat{\lambda}$ and $\hat{\nu} = \hat{\nu}$. Equality (3.1) follows. Since $M(\hat{u}) > q$ and $J'_q = 0$, (3.2) holds. Suppose $q = w$. With $\hat{\nu} = 0$, we have by (3.21) that $\hat{\nu} = c^\top(a - r1)$ and by (2.7) and (3.6), $M(\hat{u}) = w$. Since $J_w = J'_w = 0$, (3.1) and (3.2) follow. Suppose that either $q > w$ and $p(1) \neq 0$ or $p(1) = 0$ and $w < q < w + \|\beta(2)\|^2$. The above colinearity relations and the fact that $\hat{\nu} > \|p(1)\|$ if $p(1) \neq 0$ imply that $\hat{\nu} \cdot N(\hat{u}) > 0$ and that $\hat{\nu} = \hat{\lambda} N(\hat{u})$ with $\hat{\lambda} \in (0,1)$. If $p(1) = 0$, $\hat{\nu} = w + \|\beta(2)\|^2$, and $b \neq 0$, then $N(\hat{u}) = \hat{\nu}(1) - p(2) = \hat{\nu}(1) - \beta(2)$. Since $\hat{\nu}(2)$ is given by (3.19) where $\hat{z}$ is a solution of (3.17), we have that $\hat{\nu}(2) = -\beta(2)$. Thus, $\hat{\nu} = N(\hat{u})$. If $p(1) = 0$, $\hat{\nu} = w + \|\beta(2)\|^2$, and $b = 0$, then $\hat{\nu}(1) = 0$ and $\hat{\nu}(2) = (\beta(2)/\|\beta(2)\|^2)(w - q)$, so $\hat{\nu} = ((q - w)/\|\beta(2)\|^2)N(\hat{u})$. Thus, if $q > w$, then $\hat{\nu} = \hat{\lambda} N(\hat{u})$ with $\lambda > 0$ and $\hat{\nu} \cdot N(\hat{u}) > 0$. As a consequence, $M(\hat{u}) < q$. An argument similar to the one for $q < w$ shows that (3.2) holds. Since $J_q = 0$, (3.1) holds too.

Let us now consider the case where $\beta$ is in the range of $b^T$. If $q < w - \|p\|^2/2$, then on taking $\hat{\nu} = c^\top(a - r1)$, we have by the equality $M(c^\top(a - r1)) = w$ (see (3.9)) and the fact that $J'_q = 0$ that (3.2) holds. For $w - \|p\|^2/2 \leq q < w$, we define $\hat{\nu}$ by (3.22) where $\hat{\lambda} \in (0,1]$ and satisfies (3.23). We also define $\hat{\nu}$ by (3.21). We then have that $N(\hat{u}) = \hat{\nu} - p = (\lambda - 1)/\hat{\lambda} \hat{\nu}$. If, moreover, $w - \|p\|^2/2 = q$, then $\lambda = 1$, hence, $N(\hat{u}) = 0$, and by (3.20), $M(\hat{u}) = q$. Therefore, the righthand sides of (3.1) and (3.2) equal zero. As a consequence, (3.2) holds. On the other hand, $J_q = \|p\|^2/2$, so (3.1) holds only when $p = 0$. If $w - \|p\|^2/2 < q < w$, then $\lambda < 1$, hence, $N(\hat{u}) \cdot \hat{\nu} < 0$. Therefore, $M(\hat{u}) > q$ and we deduce in analogy with an earlier argument that the righthand side of (3.1) attains infimum at $\hat{\nu}$. Thus, (3.1) holds. It also holds when $q = w$ as in that case one can take $\hat{\nu} = 0$. A similar conclusion applies to (3.2). Suppose $q > w$. If $p \neq 0$, we define $\hat{\nu}, \hat{\lambda},$ and $\hat{\nu}$ by (3.22), (3.23), and by (3.21), respectively, and by the condition that $\lambda < 0$. Since $N(\hat{u}) = (\lambda - 1)p$, we have that $\hat{\nu} = \hat{\lambda} N(\hat{u})$ with $\hat{\lambda} \in (0,1)$. Then, $N(\hat{u}) \cdot \hat{\nu} > 0$, so by (3.20), $M(\hat{u}) < q$. As before, since $N(\hat{u})$ and $\hat{\nu}$ are colinear, (3.2) holds. If $p = 0$ and $b \neq 0$, then on taking as $\hat{\nu}$ a vector from the range of $b^T$ with $\|\hat{\nu}\|^2 = 2(q - w)$ and defining $\hat{\nu}$ by (3.21), we have that $N(\hat{u}) = \hat{\nu}, \hat{\nu}(1) \cdot \hat{\nu} > 0$, $M(\hat{u}) < q$ and (3.2) holds. As $J_q = 0$ when $q > w,$ (3.1) holds.

Finally, if $a - r1$ is not in the range of $b$, then by considering $u$ being a scalar multiple of the projection of $a - r1$ onto the nullspace of $b^T$ we conclude that $\text{sup}_{v \in \mathbb{R}^m}\{M(u) + N(u) \cdot v\} = \infty$ for all $v \in \mathbb{R}^m$. Hence, the constraint set in (3.1) is empty and $J_q = \infty$. Accordingly, $J'_q = 0$. We have (3.2) with $\hat{\nu}$ being any vector such that $M(\hat{u}) \geq q$.

We summarise our findings in the following lemma.
Lemma 3.1. Given \( q \in \mathbb{R} \), one can define vector \( \hat{\upsilon} \) with the following properties:

if \( J_{q^-} < \infty \), then

\[
J_{q} = \inf \left( \frac{1}{2} \|v\|^2; \; M(\hat{\upsilon}) + N(\hat{\upsilon}) \cdot v \leq q \right),
\]

if \( J_q' < \infty \), then

\[
J'_q = \inf \left( \frac{1}{2} \|v\|^2; \; M(\hat{\upsilon}) + N(\hat{\upsilon}) \cdot v \geq q \right),
\]

and if \( J_{q^+} < \infty \), then

\[
J_{q^+} = \inf \left( \frac{1}{2} \|v\|^2; \; M(\hat{\upsilon}) + N(\hat{\upsilon}) \cdot v > q \right).
\]

If \( a - r \mathbf{1} \) is in the range of \( b \), \( q > w \), and \( J_q' < \infty \), then there exists a nonzero vector \( \hat{v} \in \mathbb{R}^n \) such that \( J_q' = \|\hat{v}\|^2/2 \), (3.20) and (3.21) hold, and \( \hat{\upsilon} = \hat{\lambda}N(\hat{\upsilon}) \) where \( \hat{\lambda} > 0 \). In addition, \( \hat{\lambda} < 1 \) provided \( p^{(1)} \neq 0 \) or \( p^{(1)} = 0 \) and \( w < q < w + \|\beta(2)\|^2 \), \( \hat{\lambda} = 1 \) provided \( p^{(1)} = 0 \), \( q \geq w + \|\beta(2)\|^2 \), and \( b \neq 0 \), and \( \hat{\lambda} > 1 \) provided \( p^{(1)} = 0 \), \( \beta(2) \neq 0 \), \( q \geq w + \|\beta(2)\|^2 \), and \( b = 0 \).

Also, the following properties hold. We have that \( J_q = \infty \) if and only if either \( a - r \mathbf{1} \) is in the range of \( b \), \( \beta(2) = 0 \) and \( q < w - \|p\|^2/2 \), or \( a - r \mathbf{1} \) is not the range of \( b \), and we have that \( J_q' = \infty \).

\[
\text{if and only if } b = 0, \beta = 0, a - r \mathbf{1} = 0, \text{ and } q > w.
\]

The function \( J_q \) is rightcontinuous and the function \( J_q' \) is leftcontinuous. If \( J_{q^-} < \infty \), then \( J_{q^-} = J_q \) and if \( J_{q^+} < \infty \), then \( J_{q^+} = J_q' \). The equality \( J_q = J_{q^-} \) is violated in the only case where \( a - r \mathbf{1} \) is in the range of \( b \), \( \beta \) is in the range of \( b^T \), and \( q = w - \|p\|^2/2 \). The equality \( J_q = J_{q^+} \) is violated in the only case where \( b = 0, \beta = 0, a - r \mathbf{1} = 0, \text{ and } q = r - \alpha + \gamma/2 \). If \( a - r \mathbf{1} \) is in the range of \( b \) and \( \beta \) is not in the range of \( b^T \), \( q > w - \|p\|^2/2 \), and either \( p \neq 0 \) or \( b \neq 0 \), then \( J_{q^-} < \infty \) and \( J_{q^+} < \infty \). If \( a - r \mathbf{1} \) is in the range of \( b \), then \( J_q = 0 \) if and only if \( q \geq w \) and \( J_q' = \infty \) if and only if \( q \leq w \).

Proof. The only claim that needs substantiating is that if \( J_{q^+} < \infty \), then

\[
J_{q^+} = \inf \left( \frac{1}{2} \|v\|^2; \; M(\hat{\upsilon}) + N(\hat{\upsilon}) \cdot v > q \right).
\]

It follows by the inequalities (note that \( J_{q^+\epsilon} < \infty \) for all \( \epsilon > 0 \) small enough)

\[
\inf \left( \frac{1}{2} \|v\|^2; \; M(\hat{\upsilon}) + N(\hat{\upsilon}) \cdot v \geq q \right) \leq \inf \left( \frac{1}{2} \|v\|^2; \; M(\hat{\upsilon}) + N(\hat{\upsilon}) \cdot v > q \right) \leq J_{q^+}
\]

and the fact that the leftmost side and the rightmost side both equal \( J_q' \) when \( J_{q^+} < \infty \). \( \square \)

In the proofs of Theorems 2.1 and 2.2 we will also need the following lemma.

Lemma 3.2. The following representation holds

\[
J_q = \sup_{u \in \mathbb{R}^n} \inf \left( \frac{1}{2} \|v\|^2; \; M(u) + N(u) \cdot v \leq q \right).
\]

In addition, if \( J_q' < \infty \), then there exists \( \lambda \in \mathbb{R}^n \) such that

\[
J_q' = \lambda q - \sup_{u \in \mathbb{R}^n} \left( \lambda M(u) + \frac{\lambda^2}{2} \|N(u)\|^2 \right).
\]
Proof. We denote 
\[ \tilde{J}_q = \sup_{u \in \mathbb{R}^n} \inf \left\{ \frac{1}{2} \|v\|^2 ; M(u) + N(u) \cdot v \leq q \right\}. \]

By (3.3), \( J_q \geq \tilde{J}_q \). In order to prove the opposite inequality, suppose \( J_q < \infty \). Then \( J_{q+\epsilon} < \infty \) for all \( \epsilon > 0 \), in particular, \( J_{(q+\epsilon)^-} < \infty \). By Lemma 3.1, there exist \( \hat{u}_\epsilon \) such that \( J_{q+\epsilon}^\circ = \inf (\|v\|^2/2 ; M(\hat{u}_\epsilon) + N(\hat{u}_\epsilon) \cdot v \leq q + \epsilon) \). Therefore, \( J_{q+\epsilon} = \tilde{J}_{q+\epsilon} \). As \( \epsilon \to 0 \), the \( \tilde{J}_{q+\epsilon} \) monotonically increase to \( \tilde{J}_q \), also \( J_{q+\epsilon} \uparrow J_q \). Thus, \( J_q = \tilde{J}_q \). Suppose \( J_q = \infty \). If \( a - r1 \) is in the range of \( b, \beta(2) = 0 \), and \( q < w - \|p\|^2/2 \), then \( M(c^1 b^\beta) = w - \|p\|^2/2 \) and \( N(c^1 b^\beta) = 0 \). Therefore, \( M(c^1 b^\beta) + N(c^1 b^\beta) \cdot v > q \) for all \( v \), so \( \tilde{J}_q = \infty \). If \( a - r1 \) is not in the range of \( b \), then on taking as \( u \) a suitable vector from the nullspace of \( b^T \), we can make the infimum of \( \|v\|^2 \) over the set \( M(u) + N(u) \cdot v \leq q \) arbitrarily large.

We prove the second part. Let \( \overline{\lambda} = 0 \) if either \( q \leq w \) or \( a - r1 \) is not in the range of \( b \), and let \( \overline{\lambda} = \lambda \) otherwise, where \( \lambda \) satisfies the requirements of Lemma 3.1. We only need to consider the case where \( q > w \) and \( a - r1 \) is in the range of \( b \). Let us prove that

\[ \sup_{u \in \mathbb{R}^n} (\overline{\lambda} M(u) + \overline{\lambda}^2 \|N(u)\|^2) = \overline{\lambda} M(\hat{u}) + \overline{\lambda}^2 \|N(\hat{u})\|^2. \] (3.28)

Suppose, \( \overline{\lambda} < 1 \). Since \( a - r1 \) is in the range of \( b \), by (2.7) and (2.8) the supremum on the left of (3.28) is attained at

\[ \hat{u} = \frac{1}{1 - \overline{\lambda}} c^1 (a - r1 - \overline{\lambda} b^\beta). \]

It is not difficult to see by using the definition of \( p \) in (3.5), the definition of \( \hat{u} \) in (3.21), and the relations \( \hat{v} = \overline{\lambda} N(\hat{u}) \) and \( N(\hat{u}) = \hat{v}^{(1)} - p \), that \( \hat{u} = \hat{u} \).

If \( \overline{\lambda} = 1 \) so that \( p^{(1)} = 0 \) by Lemma 3.1, then by (2.7), (2.8), and (3.5), \( \overline{\lambda} M(u) + (\overline{\lambda}^2/2) \|N(u)\|^2 = r - \alpha + \gamma/2 + \|\beta\|^2/2 \) for all \( u \in \mathbb{R}^n \), so (3.28) also holds. Finally, if \( \overline{\lambda} > 1 \), then by Lemma 3.1, (2.7), and (2.8), \( M(u) = r - \alpha + \gamma/2 \) and \( N(u) = -\beta \) for all \( u \in \mathbb{R}^n \) and we arrive at (3.28).

By (3.28), on recalling (3.20),

\[ \overline{\lambda} q - \sup_{u \in \mathbb{R}^n} \left( \overline{\lambda} M(u) + \overline{\lambda}^2 \|N(u)\|^2 \right) \]

which equals \( J_q' \) by Lemma 3.1.

\[ \square \]

4 Proof of Theorem 2.1

Let us prove that

\[ \liminf_{t \to \infty} \inf_{\pi \in \mathcal{U}} \frac{1}{t} \ln P(L_t^\pi < q) \geq -J_{q^-}. \] (4.1)

We may assume that \( J_{q^-} < \infty \). Hence, there exists \( v \in \mathbb{R}^m \) with

\[ \sup_{u \in \mathbb{R}^n} (M(u) + N(u) \cdot v) < q. \] (4.2)

We use it in order to change the probability measure. Let \( W'_s \) for \( s \in [0,1] \) and measure \( P^* \) be defined by the respective equalities

\[ W'_s = \sqrt{t} v s + W'_s \] (4.3)
and

$$\frac{dP}{dP^*} = \exp\left(-\sqrt{t} \cdot v \cdot W^*_1 - \frac{t}{2} \|v\|^2\right). \quad (4.4)$$

Then by (2.9)

$$P(L_{t}^\pi < q) = E^* 1_{\{\ln Z_0/t + \int_0^t M(\pi_s) \cdot v \, ds + (1/\sqrt{t}) \int_0^t N(\pi_s) \cdot dW^*_s < q\}} \exp\left(-\sqrt{t} \cdot v \cdot W^*_1 - \frac{t}{2} \|v\|^2\right),$$

where $1_{\Gamma}$ denotes the indicator function of event $\Gamma$. By Girsanov’s theorem, the process $(W^*_s, s \in [0,1])$ is a standard Wiener process under $P^*$. By (4.2), there exists $\epsilon > 0$ such that for all $\pi_t$

$$\int_0^1 M(\pi_s) \, ds + \int_0^1 N(\pi_s) \cdot v \, ds < q - \epsilon,$$

so, for arbitrary $\delta > 0$,

$$P(L_{t}^\pi < q) \leq E^* 1_{\{\ln Z_0/t \geq \epsilon/2\}} \frac{1}{\delta^2} \exp\left(-\frac{t}{2} \|v\|^2\right),$$

we have that

$$\lim_{t \to \infty} P^*\left(\frac{1}{\sqrt{t}} v \cdot W^*_1 > \frac{\delta}{\epsilon^2}\right) = 1. \quad (4.6)$$

Next, for arbitrary $\eta > 0$, by the Lenglart-Rebolledo inequality,

$$P^*\left(\frac{\ln Z_0}{t} + \int_0^1 N(\pi_s) \cdot dW^*_s > \epsilon\right) \leq P^*\left(\frac{\ln Z_0}{t} \geq \frac{\epsilon}{2}\right) + \frac{4\eta}{\epsilon^2} + P^*\left(\int_0^1 \|N(\pi_s)\|^2 \, ds > \eta\right). \quad (4.7)$$

By (4.3), (4.4), and the Cauchy-Schwartz inequality,

$$P^*\left(\frac{\ln Z_0}{t} \geq \frac{\epsilon}{2}\right) = E^* 1_{\{\ln Z_0/t \geq \epsilon/2\}} \exp\left(\sqrt{t} \cdot v \cdot W_1 - \frac{t}{2} \|v\|^2\right) \leq P\left(\frac{\ln Z_0}{t} \geq \frac{\epsilon}{2}\right)^{1/2} \exp\left(\frac{t\|v\|^2}{2}\right),$$

so by (2.13)

$$\lim_{t \to \infty} P^*\left(\frac{\ln Z_0}{t} \geq \frac{\epsilon}{2}\right)^{1/2} = 0,$$

which implies that

$$\lim_{t \to \infty} P^*\left(\frac{\ln Z_0}{t} \geq \frac{\epsilon}{2}\right) = 0.$$

By a similar argument, in view of (2.12),

$$\lim \sup_{t \to \infty} P^*\left(\frac{1}{t} \int_0^1 \|N(\pi_s)\|^2 \, ds > \eta\right) = 0.$$
Thus, in view of (4.7)

$$\lim_{t \to \infty} \inf_{\pi \in U} P^*(\frac{\ln Z_0}{t} + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_s^t) \cdot dW_s^* \leq \epsilon) = 1.$$ 

By (4.6), we have that

$$\lim_{t \to \infty} \inf_{\pi \in U} P^*(\frac{1}{\sqrt{t}} v \cdot W_1^* \leq \delta, \frac{\ln Z_0}{t} + \frac{1}{\sqrt{t}} \int_0^1 N(\pi_s^t) \cdot dW_s^* \leq \epsilon) = 1.$$ 

It follows by (4.5) that

$$\liminf_{t \to \infty} \inf_{\pi \in U} P(L_\pi^t \leq q)^{1/t} \geq \exp(-\delta - \frac{1}{2} \|v\|^2).$$

Taking on the right the supremum over $\delta > 0$ and over $v$ satisfying (4.2) and recalling (3.3) yields (4.1).

We prove the upper bound. On taking $\pi_t = u$, where $u$ is an arbitrary $n$-vector, we obtain, by (2.9), for $t > 1$ and $\epsilon > 0$,

$$P(L_u^t \leq q)^{1/t} = P(\frac{\ln Z_0}{t} + M(u) + N(u) \cdot \frac{W_1}{\sqrt{t}} \leq q)^{1/t} \leq P(\frac{\ln Z_0}{t} < -\epsilon)^{1/t} + P(M(u) + N(u) \cdot \frac{W_1}{\sqrt{t}} \leq q + \epsilon)^{1/t}.$$ 

Since $\lim_{t \to \infty} P((1/t) \ln Z_0 < -\epsilon)^{1/t} = 0$ and since by the fact that $W_1$ is standard normal

$$\limsup_{t \to \infty} \frac{1}{t} \ln P(M(u) + N(u) \cdot \frac{W_1}{\sqrt{t}} \leq q + \epsilon) \leq -\inf(\frac{1}{2} \|v\|^2; M(u) + N(u) \cdot v \leq q + \epsilon),$$

we have that

$$\limsup_{t \to \infty} \frac{1}{t} \ln \inf_{\pi \in U} P(L_u^t \leq q) \leq -\inf(\frac{1}{2} \|v\|^2; M(u) + N(u) \cdot v \leq q + \epsilon),$$

so, on letting $\epsilon \to 0$,

$$\limsup_{t \to \infty} \frac{1}{t} \ln \inf_{\pi \in U} P(L_u^t \leq q) \leq -\inf(\frac{1}{2} \|v\|^2; M(u) + N(u) \cdot v \leq q).$$

On minimising the righthand side over $u \in \mathbb{R}^n$ and invoking Lemma 3.2, we obtain the needed inequality

$$\limsup_{t \to \infty} \frac{1}{t} \ln \inf_{\pi \in U} P(L_u^t \leq q) \leq -J_q.$$ 

If, in addition, $J_q < \infty$ and $\hat{\pi}_t = \hat{u}$ where $\hat{u}$ is as in Lemma 3.1, then the above argument shows that

$$\limsup_{t \to \infty} \frac{1}{t} \ln P(L_{\hat{u}}^t \leq q) \leq -J_q,$$

so $\hat{\pi}$ is an asymptotically optimal control.
5 Proof of Theorem 2.2

The proof of Theorem 2.2 is quite different from the proof of Theorem 2.1. As is common in large deviation arguments, one uses an exponential Markov inequality. We first dispense with the easy case. If $J_\epsilon' = \infty$ so that $\beta = 0$, $b = 0$, $a - r 1 = 0$, and $q > r - \alpha + \gamma / 2$, we have that

$$P(L_t^\pi \geq q) = P\left( \ln Z_0 / t \geq q - r + \alpha - \gamma / 2 \right)$$

and the required bound follows by (2.13).

We can thus assume that $J_\epsilon' < \infty$. By leftcontinuity, $J_\epsilon' - \epsilon / 2 < \infty$ for all $\epsilon > 0$ small enough. Let $\lambda$ be chosen as in the statement of Lemma 3.2 with $q$ replaced by $q - \epsilon / 2$. We have, for $\epsilon > 0$ and $\pi \in \mathcal{U}$, by (2.9),

$$P(L_t^\pi \geq q) \leq P\left( \frac{\ln Z_0}{t} \geq \frac{\epsilon}{2} \right) + P\left( \int_0^t M(\pi_s^t) \, ds + \frac{1}{\sqrt{t}} \int_0^t N(\pi_s^t) \cdot dW_s \geq q - \frac{\epsilon}{2} \right)$$

$$\leq P\left( \frac{\ln Z_0}{t} \geq \frac{\epsilon}{2} \right) + \exp\left(-\lambda t (q - \frac{\epsilon}{2})\right) \mathbb{E} \exp\left(\lambda t \int_0^t M(\pi_s^t) \, ds + \lambda \sqrt{t} \int_0^t N(\pi_s^t) \cdot dW_s\right)$$

$$\leq P\left( \frac{\ln Z_0}{t} \geq \frac{\epsilon}{2} \right) + \exp\left(-t (\lambda(q - \frac{\epsilon}{2}) - \sup_{u \in \mathbb{R}^n} (\lambda M(u) + \frac{\lambda^2}{2} \|N(u)\|^2)\right)$$

$$\mathbb{E} \exp\left(\lambda \sqrt{t} \int_0^1 N(\pi_s^t) \cdot dW_s - \frac{\lambda^2 t}{2} \int_0^1 \|N(\pi_s^t)\|^2 \, ds\right).$$

On noting that $\mathbb{E} \exp(\lambda \sqrt{t} \int_0^1 N(\pi_s^t) \cdot dW_s - \lambda^2 t / 2 \int_0^1 \|N(\pi_s^t)\|^2 \, ds) \leq 1$, we obtain by the second part of Lemma 3.2 that for $t > 1$

$$\sup_{\pi \in \mathcal{U}} P(L_t^\pi \geq q)^{1/t} \leq P\left( \frac{\ln Z_0}{t} \geq \frac{\epsilon}{2} \right)^{1/t} + \exp(-J_\epsilon' - \epsilon / 2).$$

By (2.13), $\lim_{t \to \infty} P\left( \ln Z_0 / t \geq \epsilon / 2 \right)^{1/t} = 0$ and by Lemma 3.1, $\lim_{t \to 0} J_\epsilon' - \epsilon / 2 = J_\epsilon'$. It follows that

$$\limsup_{t \to \infty} \frac{1}{t} \ln \sup_{\pi \in \mathcal{U}} P(L_t^\pi \geq q) \leq -J_\epsilon'.$$

The bound

$$\liminf_{t \to \infty} \frac{1}{t} \ln \sup_{\pi \in \mathcal{U}} P(L_t^\pi > q) \geq -J_\epsilon'$$

is clearly true when $J_\epsilon' = \infty$. Suppose $J_\epsilon' < \infty$ and define $\hat{\pi}_t = \hat{u}$, where $\hat{u}$ is from Lemma 3.1. Then, for $t > 1$,

$$P(L_t^\pi > q)^{1/t} \geq P\left( M(\hat{u}) + N(\hat{u}) \cdot \frac{W_1}{\sqrt{t}} > q + \epsilon \right)^{1/t} - P\left( \frac{\ln Z_0}{t} \leq -\epsilon \right)^{1/t}.$$

By (2.13), $\lim_{t \to \infty} P\left( \ln Z_0 / t \leq -\epsilon \right)^{1/t} = 0$. Since $W_1$ is standard normal,

$$\liminf_{t \to \infty} \frac{1}{t} \ln P\left( M(\hat{u}) + N(\hat{u}) \cdot \frac{W_1}{\sqrt{t}} > q + \epsilon \right)^{1/t} \geq -\inf\left( \frac{1}{2} \|v\|^2; M(\hat{u}) + N(\hat{u}) \cdot v > q + \epsilon \right).$$

It follows that

$$\liminf_{t \to \infty} \frac{1}{t} \ln P(L_t^\pi > q) \geq -\inf\left( \frac{1}{2} \|v\|^2; M(\hat{u}) + N(\hat{u}) \cdot v > q \right).$$
Since $J'_q < \infty$, the latter righthand side equals $J'_q$ by Lemma 3.1. Hence,

$$\lim \inf_{t \to \infty} \frac{1}{t} \ln P(L_t^q > q) \geq -J'_q.$$ 

We conclude that if $J'_q < \infty$, then

$$\lim_{t \to \infty} \frac{1}{t} \ln P(L_t^q > q) = \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^q \geq q) = -J'_q.$$ 

6 Concluding remarks

We give a synopsis of the procedure for choosing an optimal portfolio, provide some additional comments, and consider illustrative examples. There are three distinct cases:

1. $a - r 1$ is not in the range of $b$,
2. $a - r 1$ is in the range of $b$, $\beta$ is in the range of $b^T$, and either $q \leq w - \|p\|^2/2$ or $p = 0$ and $b = 0$,
3. $a - r 1$ is in the range of $b$ and either $\beta$ is not in the range of $b^T$, or $\beta$ is in the range of $b^T$, $q > w - \|p\|^2/2$, and either $p \neq 0$ or $b \neq 0$.

We recall that $p$ is defined by (3.5) and $w$ is defined by (3.6).

If $a - r 1$ is not in the range of $b$ then there exists vector $u$ with $u \cdot (a - r 1) > 0$ which is in the nullspace of $b^T$. Then the portfolio $\pi_t = Ku$, where $K$ is an arbitrary constant, yields by (2.6)

$$L_t^\pi = \frac{\ln Z_0}{t} + Ku \cdot (a - r 1) + r - \alpha + \frac{1}{2} \gamma + \frac{1}{t} \beta \cdot W_t.$$ 

Since $\alpha > 0$, $\gamma > 0$, and $\beta > 0$, it follows that $\pi_t$ is an optimal portfolio. In this case there is an arbitrage available.

The latter means, loosely speaking, that all randomness in $I_t$ is captured by the $S_l$ which is the case, for instance, when $I_t$ is deterministic. For $\pi_t = c^\dagger (a - r 1 + bp) = c^\dagger b \beta$ by (2.6),

$$L_t^\pi = \frac{\ln Z_0}{t} + w - \frac{1}{2} \|p\|^2.$$ 

Since $J' = J'_q < \infty$, the latter righthand side equals $J'_q$ by Lemma 3.1. Hence,

$$\lim \inf_{t \to \infty} \frac{1}{t} \ln P(L_t^q > q) \geq -J'_q.$$ 

We conclude that if $J'_q < \infty$, then

$$\lim_{t \to \infty} \frac{1}{t} \ln P(L_t^q > q) = \lim_{t \to \infty} \frac{1}{t} \ln P(L_t^q \geq q) = -J'_q.$$
It has a unique positive solution \( \hat{z} \) provided that either \( \beta(2) \neq 0 \) and \( p^{(1)} \neq 0 \), or \( \beta(2) \neq 0 \) and \( q > w + \|\beta(2)\|^2 \), or \( \beta(2) = 0 \) and \( q > w - \|p^{(1)}\|^2/2 \). We let \( \hat{v}^{(1)} = (1 - \hat{z}/\|p^{(1)}\|)p^{(1)} \) if \( p^{(1)} \neq 0 \) and either \( \beta(2) \neq 0 \) or \( q > w - \|p^{(1)}\|^2/2 \). If \( p^{(1)} = 0 \), \( \beta(2) \neq 0 \), and either \( q \leq w + \|\beta(2)\|^2 \), or \( q > w + \|\beta(2)\|^2 \) and \( b = 0 \) then \( \hat{v}^{(1)} = 0 \). If \( p^{(1)} = 0 \), \( q > w + \|\beta(2)\|^2 \), and \( b \neq 0 \), we let \( \hat{v}^{(1)} \) represent any vector of magnitude \( \hat{z} \) from the range of \( b^T \). Let also

\[
\hat{v}^{(2)} = \begin{cases} 
\frac{\beta(2)}{\|\beta(2)\|^2} \left( \frac{1}{2} \|\hat{v}^{(1)}\|^2 - p^{(1)} \cdot \hat{v}^{(1)} + w - q \right), & \text{if } \beta(2) \neq 0, \\
0, & \text{otherwise.}
\end{cases}
\]

If \( q \leq w \), then the shortfall probability decay rate

\[
J_q = \frac{1}{2} \|\hat{v}^{(1)}\|^2 + \frac{1}{2} \|\hat{v}^{(2)}\|^2
\]

and the outperformance probability decay rate \( J'_q = 0 \). If \( q \geq w \), then \( J_q = 0 \) and

\[
J'_q = \frac{1}{2} \|\hat{v}^{(1)}\|^2 + \frac{1}{2} \|\hat{v}^{(2)}\|^2.
\]

If \( \hat{v}^{(1)} = (1 - \hat{z}/\|p^{(1)}\|)p^{(1)} \), then (6.1) takes the form

\[
\hat{\pi}_t = c^T(\mu(a - r1) + (1 - \mu)b\beta),
\]

where \( \mu = \hat{z}/\|p^{(1)}\| \) is the positive root of

\[
\|\beta(2)\|^2(1 - \frac{1}{\mu}) + \frac{\|p^{(1)}\|^2}{2} (\mu^2 - 1) - q + w = 0.
\]

As one can see, \( q \leq w \), respectively, \( q < w \), if and only if \( \mu \leq 1 \), respectively, \( \mu < 1 \). The decay rates \( J_q \) (or \( J'_q \)) take the form

\[
\frac{1}{2} (1 - \mu)^2\|p^{(1)}\|^2 + \frac{1}{2} (1 - \frac{1}{\mu})^2\|\beta(2)\|^2.
\]

Let us consider the example of one risky security with the Wiener processes driving the risky security and the benchmark being correlated. Thus,

\[
dS_t = S_t a dt + S_t b dW_t^{(1)},
\]

\[
 dB_t = B_t r dt,
\]

\[
 dI_t = I_t \alpha dt + I_t \beta dW_t^{(2)},
\]

where \( W_t^{(1)} \) and \( W_t^{(2)} \) are standard one-dimensional Wiener processes with \( \mathbb{E} W_t^{(1)} W_t^{(2)} = \rho t \). Since we can take \( W_t^{(1)} = W_t^{(3)} \sqrt{1 - \rho^2} + W_t^{(2)} \rho \), where \( W_t^{(3)} \) is a standard Wiener process independent of \( W_t^{(2)} \), the equations are brought to the canonical form as follows

\[
 dS_t = S_t a dt + S_t b \left[ \rho \sqrt{1 - \rho^2} \right] \begin{bmatrix} dW_t^{(2)} \\ dW_t^{(3)} \end{bmatrix},
\]

\[
 dB_t = B_t r dt,
\]

\[
 dI_t = I_t \alpha dt + I_t \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} dW_t^{(2)} \\ dW_t^{(3)} \end{bmatrix}.
\]
We assume that $b > 0$, so $a - r$ is in the range of $b [\rho \sqrt{1 - \rho^2}]$. The vector $\beta [1 0]^T$ plays the role of the vector $\beta$ in the general model. By (3.6),

$$w = \frac{(a - r)^2}{2b^2} + r - \alpha + \frac{\beta^2}{2}.$$ 

By (3.5),

$$p = \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{a - r}{b} \begin{bmatrix} \rho \\ \sqrt{1 - \rho^2} \end{bmatrix}.$$ 

Therefore, $\|u^{(1)}\|^2 = (\beta \rho - (a - r)/b)^2$ and $\|u^{(2)}\|^2 = \beta^2 (1 - \rho^2)$. The equation for $\mu$ is

$$\mu^2 \left( \beta \rho - \frac{a - r}{b} \right)^2 - 2 \left( \frac{1}{\mu} - 1 \right) \beta^2 (1 - \rho^2) = 2(q - w) + \left( \beta \rho - \frac{a - r}{b} \right)^2. \quad (6.3)$$

The optimal control from (6.2) is

$$\hat{\pi}_t = \frac{1}{b^2} \left( \mu(a - r) + (1 - \mu) \beta \rho \right). \quad (6.4)$$

The associated decay rates are

$$J_q = \frac{1}{2} \left( 1 - \min(1, \mu) \right)^2 \left( \beta \rho - \frac{a - r}{b} \right)^2 + \frac{1}{2} \left( 1 - \max(1, \mu^{-1}) \right)^2 \beta^2 (1 - \rho^2), \quad (6.5a)$$

$$J_q' = \frac{1}{2} \left( 1 - \max(1, \mu) \right)^2 \left( \beta \rho - \frac{a - r}{b} \right)^2 + \frac{1}{2} \left( 1 - \min(1, \mu^{-1}) \right)^2 \beta^2 (1 - \rho^2). \quad (6.5b)$$

Consider the case of the Black-Scholes model and a constant benchmark. We thus take $\alpha = \beta = 0$, so $w - \|p^{(1)}\|^2/2 = r$. For $q \leq r$, there exists a perfectly outperforming portfolio when all wealth is invested in bond (that is case 2 above). Suppose $q > r$. Then, on assuming $a > r$, we are justified in using (6.3), (6.4), (6.5a), and (6.5b). We have

$$\mu = \frac{b}{a - r} \sqrt{2(q - r)},$$

so the share of the wealth invested in stock is

$$\hat{\pi}_t = \frac{1}{b} \sqrt{2(q - r)}.$$ 

The decay rates are

$$J_q = \frac{1}{2} \left( \sqrt{2(q - r)} - \frac{a - r}{b} \right)^2, \quad J_q' = 0,$$

provided $r < q \leq (a - r)^2/(2b^2) + r$ and

$$J_q' = \frac{1}{2} \left( \sqrt{2(q - r)} - \frac{a - r}{b} \right)^2, \quad J_q = 0,$$

provided $q \geq (a - r)^2/(2b^2) + r$. The optimal control for $q \geq (a - r)^2/(2b^2) + r$ and $J_q'$ were found by Pham [6]. However, Pham [6] obtains a different control for $q \leq (a - r)^2/(2b^2) + r$, which is not a surprise given there are many outperforming portfolios when $J_q' = 0$.

**Acknowledgement.** The author is thankful to Michael Stutzer for illuminating discussions and to Ilya Lashuk for useful comments. Thanks are also due to the referees for their careful reading of the manuscript and for sharing their insight.
References


