A GENERAL LOWER BOUND FOR POTENTIALLY $H$-GRAPHIC SEQUENCES

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Abstract. We consider a variation of the classical Turán-type extremal problem as introduced by Erdős et al. in [7]. Let $\pi$ be an $n$-element graphic sequence, and $\sigma(\pi)$ be the sum of the terms in $\pi$, that is the degree sum. Let $H$ be a graph. We wish to determine the smallest $m$ such that any $n$-term graphic sequence $\pi$ having $\sigma(\pi) \geq m$ has some realization containing $H$ as a subgraph. Denote this value $m$ by $\sigma(H,n)$. For an arbitrarily chosen $H$, we construct a graphic sequence $\pi^*(H,n)$ such that $\sigma(\pi^*(H,n)) + 2 \leq \sigma(H,n)$. Furthermore, we conjecture that equality holds in general, as this is the case for all choices of $H$ where $\sigma(H,n)$ is currently known. We support this conjecture by examining those graphs that are the complement of triangle-free graphs, and showing that the conjecture holds despite the wide variety of structure in this class. We will conclude with a brief discussion of a connection between potentially $H$-graphic sequences and $H$-saturated graphs of minimum size.

Keywords: Degree sequence, Potentially graphic sequence, $H$-saturated graph.

1. Introduction

A good reference for any undefined terms is [1]. Let $G$ be a simple undirected graph, without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$ respectively and let $d(v)$ denote the degree of a vertex $v$. Let $\bar{G}$ denote the complement of $G$. Denote the complete graph on $t$ vertices and the complete bipartite graph with partite sets of size $r$ and $s$ by $K_t$ and $K_{r,s}$ respectively. Additionally, let $K_s^t$ denote the complete balanced multipartite graph with $t$ partite sets of size $s$. Given any two graphs $G$ and $H$, their join, denoted $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{gh \mid g \in V(G), h \in V(H)\}$. Additionally, let $\alpha(G)$ denote the independence number of $G$. If $H$ is a subgraph of $G$, we will write $H \subseteq G$, and if $H$ is an induced subgraph of $G$, we will write $H < G$.

A sequence of nonnegative integers $\pi = (d_1, d_2, ..., d_n)$ is called graphic if there is a (simple) graph $G$ of order $n$ having degree sequence $\pi$. In this case, $G$ is said
to realize $\pi$, and we will write $\pi = \pi(G)$. If a sequence $\pi$ consists of the terms $d_1, \ldots, d_t$ having multiplicities $\mu_1, \ldots, \mu_t$, we may write $\pi = (d_1^{\mu_1}, \ldots, d_t^{\mu_t})$.

For a given graph $H$, a sequence $\pi$ is said to be potentially $H$-graphic if there is some realization of $\pi$ which contains $H$ as a subgraph. Additionally, let $\sigma(\pi)$ denote the sum of the terms of $\pi$. Define $\sigma(H, n)$ to be the smallest integer $m$ so that every $n$-term graphic sequence $\pi$ with $\sigma(\pi) \geq m$ is potentially $H$-graphic. In this paper, given an arbitrary $H$, we construct a graphic sequence $\pi^*(H, n)$ such that $\sigma(\pi^*(H, n)) + 2 \leq \sigma(H, n)$. We then show that equality holds for all graphs $H$ that are the complement of a triangle-free graph. There have been numerous papers, including but certainly not limited to [5], [3], [4], [7], [9], [11], [12], [14], [15], [16], [17] and [18], that consider the potential problem for specific graphs or narrow families of graphs. It is our hope that the ideas and results presented in this paper will facilitate a broader consideration of problems of this type.

2. A Short History

In this section, we present the extremal sequences for two classes of graphs: complete graphs and complete balanced bipartite graphs. Our goal is to motivate the general constructions in the next section.

2.1. $H = K_t$. In [7] Erdős, Jacobson and Lehel conjectured that $\sigma(K_t, n) = (t - 2)(2n - t + 1) + 2$. The conjecture rises from consideration of the graph $K_{(t-2)} + \overline{K}_{(n-t+2)}$. It is easy to observe that this graph contains no $K_t$, is the unique realization of the sequence $((n - 1)t^2 - (t - 2)n + t + 2)$, and has degree sum $(t - 2)(2n - t + 1)$. The cases $t = 3, 4$ and 5 were proved separately (see respectively [7], [12] and [15], and [16]), and Li, Song and Luo [17] proved the conjecture true via linear algebraic techniques for $t \geq 6$ and $n \geq \left(\frac{t}{2}\right) + 3$. A purely graph-theoretic proof was given in [10] and also as a corollary to the main result in [4].

2.2. $H = K_{s,s}$. The following results appear in [12] and [18]. Here $E_1, E_2, E_3$ and $E_4$ are somewhat technical numerical classes which, based on the parity of $n$ and $s$, assure that the given degree sums are even.

**Theorem 2.1.**

- If $s$ is an odd, positive integer and $n \geq 4s^2 + 3s - 8$, then

$$\sigma(K_{s,s}, n) = \begin{cases} 
\left(\frac{s}{2} - \frac{3}{2}\right)n - \frac{11}{2}s^2 + \frac{5}{2}s + 7 & \text{if } (s, n) \in E_3 \\
\left(\frac{s}{2} - \frac{5}{2}\right)n - \frac{11}{4}s^2 + \frac{5}{2}s + \frac{16}{3} & \text{if } (s, n) \in E_4.
\end{cases} \quad (1)$$

- If $s$ is an even, positive integer and $n \geq 4s^2 - s - 6$, then

$$\sigma(K_{s,s}, n) = \begin{cases} 
\left(\frac{s}{2} - 2\right)n - \frac{11}{4}s^2 + \frac{5}{2}s + 2 & \text{if } (s, n) \in E_1 \\
\left(\frac{s}{2} - 2\right)n - \frac{11}{4}s^2 + \frac{5}{2}s + 1 & \text{if } (s, n) \in E_2.
\end{cases} \quad (2)$$

In order to establish a lower bound on $\sigma(K_{s,s}, n)$ the authors present several sequences dependent on the parities of $s$ and $n$. 
(i) If \( s \) is odd and \((s, n) \in E_3\), then
\[
\pi(K_{s, s, n}) = ((n - 1)^{s-1}, 2s - 2, 2s - 3, \ldots, \frac{3}{2}s + \frac{3}{2}, \frac{3}{2}s + 1, 2s - 1),
\]
\[
(\frac{3}{2}s - \frac{1}{2})^+2, (\frac{3}{2}s - \frac{3}{2})^{n-2s}, \frac{3}{2}s - \frac{5}{2}).
\] (3)

(ii) If \( s \) is odd and \((s, n) \in E_4\), then
\[
\pi(K_{s, s, n}) = ((n - 1)^{s-1}, 2s - 2, 2s - 3, \ldots, \frac{3}{2}s + \frac{3}{2}, \frac{3}{2}s + 1, 2s - 1),
\]
\[
(\frac{3}{2}s - \frac{1}{2})^+2, (\frac{3}{2}s - \frac{3}{2})^{n-2s+1}).
\] (4)

(iii) If \( s \) is even and \((s, n) \in E_1\), then
\[
\pi(K_{s, s, n}) = ((n - 1)^{s-1}, 2s - 2, 2s - 3, \ldots, \frac{3}{2}s + 1, \frac{3}{2}s, (\frac{3}{2}s - 1)^{n-2s+2}).
\] (5)

(iv) If \( s \) is even and \((s, n) \in E_2\), then
\[
\pi(K_{s, s, n}) = ((n - 1)^{s-1}, 2s - 2, 2s - 3, \ldots, \frac{3}{2}s + 1, \frac{3}{2}s, (\frac{3}{2}s - 1)^{n-2s+1}, (\frac{3}{2}s - 2)).
\] (6)

Each of these sequences can be realized by the join of \( K_{s-1} \) and some graph \( H' \). This \( H' \) has no vertices of degree \( s \), one vertex of degree \( s - 1 \), two vertices of degree \( s - 2 \) and so on. More generally, no choice of \( H' \) contains \( x_1 \) vertices of degree \( x_2 \), where \( x_1 + x_2 = s + 1 \). This implies that \( H' \) cannot possibly contain a copy of \( K_{x_1, x_2} \). However, if any of these sequences were to be potentially \( K_{s, s} \)-graphic, at least \( s + 1 \) of the vertices in a copy of \( K_{s, s} \) would have to be chosen from \( H' \). These vertices in turn, would comprise some \( K_{x_1, x_2} \) where \( x_1 + x_2 = s + 1 \).

3. A General Lower Bound

We assume that \( H \) has no isolated vertices and furthermore that \( n \) is sufficiently large relative to \(|V(H)|\). We define the quantities
\[
u(H) = |V(H)| - \alpha(H) - 1,
\]
and
\[
d(H) = \min\{\Delta(F) : F < H, |V(F)| = \alpha(H) + 1\}.
\]

Consider the following sequence,
\[
\hat{\pi}(H, n) = ((n - 1)^{u(H)}, u(H) + d(H) - 1)^{n-u(H)}.
\] (7)

If this sequence is not graphic, that is if \( n - u(H) \) and \( d(H) - 1 \) are both odd, we reduce the smallest term by one. To see that this will result in a graphic sequence, we make two observations. First, \((d(H) - 1)-regular graphs of order \( n - u(H) \) \geq d(H) \) exist whenever \( d(H) - 1 \) and \( n - u(H) \) are not both odd. If \( n \) and \( d(H) - 1 \) are both odd, it is not difficult to show that the sequence \((d(H) - 1)^{n-u(H)} - 1, d(H) - 2)\) is graphic.

Every realization of \( \hat{\pi}(H, n) \) is a complete graph on \( u(H) \) vertices, joined to a graph, call it \( G' \), that is either \((d(H) - 1)\)-regular or nearly so. Note that the
Theorem 3.1. Given a graph $H$, with $u(H)$ and $d(H)$ as above, and $n$ sufficiently large then,

$$\sigma(H, n) \geq \max\{\sigma(H^*, n) + 2 \mid H^* \subseteq H\}. \quad (10)$$

Proof. Let $H^*$ be the subgraph of $H$ that realizes the maximum above. Let $G$ be any realization of $\pi^*(H^*, n)$. We show that $G$ does not contain a copy of $H^*$.
Note that this degree sequence implies that $G$ is a copy of $K_{u(H^*)}$ joined to another graph $G^*$ on $n - u(H^*)$ vertices. Assume that there is a copy of $H^*$ contained in $G$. There are at least $\alpha(H^*) + 1$ vertices from $G^*$ that must belong to this copy of $H$. Let $H^{**}$ denote the subgraph of $H^*$ induced by these $\alpha(H^*) + 1$ vertices. Notice, however, no $\alpha(H^*) + 1$ vertices of $G^*$ have sufficient degree to contain a copy of $H^{**}$. In particular, if $\sum_{j \geq \ell} v_j(H^{**}) > 0$ then $H^{**}$ contains at least $m_\ell$ vertices of degree $\ell$ or greater. By our construction, there are at most $n_\ell \leq m_\ell$ vertices of degree at least $\ell$ in $G^*$. This contradicts the assumption that $H^{**} \subseteq G^*$. Thus, $G$ contains no copy $H^*$ and hence no copy of $H$. □

Theorem 3.1 requires that we examine all subgraphs of $H$. To see that this is necessary, we consider the split graph $K_t + K_s$ with a pendant vertex $v$ adjacent to one of the vertices in the independent set of order $s$. For this choice of $H$, $\alpha(H) = s$ and hence $u(H) = (s + t + 1) - s - 1 = t$ and $d(H) = 1$. However, if we remove $v$, the pendant vertex, and consider the split graph, we can see that $u(K_t + K_s) = t - 1$ but any $(s + 1)$-vertex subgraph of $K_t + K_s$ must contain some vertex from the $K_t$, implying that $d(K_t + K_s) = s$. Therefore, if we choose $s \geq 3$, $\sigma(\pi^*(K_t + K_s, n)) \geq \sigma(\pi^*(H, n))$.

The reader should note that for any values of $n$ and $s$, $\pi^*(K_{s,s}, n)$ is exactly those sequences given in (3)-(6). Additionally, given values of $n, s$ and $t$, $\pi^*(K^t_{s}, n)$ matches the extremal sequences given in [23].

We conjecture that equality holds in Theorem 3.1.

**Conjecture 1.** Let $H$ be any graph, and let $n$ be a sufficiently large integer. Then

$$\sigma(H, n) = \max\{\sigma(\pi^*(H^*, n)) + 2 \mid H^* \subseteq H\}. \quad (11)$$

We also pose the weaker conjecture, that the bound put forth is asymptotically correct.

**Conjecture 2.** Let $H$ be any graph, and let $\epsilon > 0$. Then there exists an $n_0 = n_0(\epsilon, H)$ such that for any $n > n_0$

$$\sigma(H, n) \leq \max\{(n(2u(H^*) + d(H^*) - 1 + \epsilon) \mid H^* \subseteq H\}. \quad (12)$$

Conjectures 1 and 2 have been verified for a wide variety of graphs. This includes, but is not limited to: complete graphs and unions of complete graphs [7], [9], [12], [15], [16], [17], complete bipartite graphs [3],[12], [18], complete multipartite graphs [5], [20], matchings [12], cycles [14], (generalized) friendship graphs [2], [9], [11], and split graphs [4]. At this time we know of no subgraph for which these conjectures do not hold for sufficiently large $n$.

While Conjecture 1 seems challenging, we feel that there is a good chance that Conjecture 2 could be verified. In the following section, we will verify Conjecture 1 for a broad class of graphs.
4. COMPLEMENTS OF TRIANGLE-FREE GRAPHS

We now turn our attention to graphs $H$ of order $k \geq 3$ with $\alpha(H) = 2$, or those graphs that are the complement of a triangle-free graph. The main result of this section is as follows.

**Theorem 4.1.** Let $H$ be any graph of order $k$ with $\alpha(H) = 2$. Then

$$\sigma(H, n) = \sigma(\pi^*(H, n)) + 2.$$

Any graph $H$ in this class has $u(H) = k - 3$ and $d(H) \leq 2$. We prove Theorem 4.1 by considering the cases $d(H) = 1$ and $d(H) = 2$ separately. In each case we construct a graph $H(d)$ that contains $H$ as a subgraph and show that $\sigma(H(d), n) = \sigma(\pi^*(H, n)) + 2$. This implies that $\max\{\sigma(\pi^*(H^*, n)) + 2 \mid H^* \subseteq H\} = \sigma(\pi^*(H, n)) + 2$.

The following result from [4] will be very useful.

**Theorem 4.2.** If $n \geq 3s + 2t^2 + 3t - 3$ then

$$\sigma(K_s + \overline{K}_t, n) = \begin{cases} (t + 2s - 3)n - (s - 1)(s + t - 1) + 2 & \text{if } t \text{ or } n - s \text{ is odd.} \\ (t + 2s - 3)n - (s - 1)(s + t - 1) + 1 & \text{if } t \text{ and } n - s \text{ are even.} \end{cases}$$

It is not difficult to see that if $d(H) = 2$ then $H$ is isomorphic to $K_k - tK_2$, where $k$ is the order of $H$ and $t$ is some positive integer that is at most $\frac{k}{2}$. Let $H$ be a graph of order $k \geq 3$ with $\alpha(H) = 2$ and $d(H) = 2$ and let $n \geq k$ be an integer. Then, by (9), we have.

(i) If $n \equiv k - 3 \pmod{2}$ then

$$\pi^*(H, n) = ((n - 1)^{k-3}, (k - 2)^{n-k+3})$$

(ii) If $n \not\equiv k - 3 \pmod{2}$ then

$$\pi^*(H, n) = ((n - 1)^{k-3}, (k - 2)^{n-k+2}, k - 3)$$

**Proposition 4.3.** Let $H$ be a graph of order $k$ with $\alpha(H) = 2$ and $d(H) = 2$, and let $n$ be a sufficiently large integer. Then

$$\sigma(H, n) = \sigma(\pi^*(H, n)) + 2 = n(2k - 5) - k^2 + 4k - 1 - m,$$

where $m = n - k + 3 \pmod{2}$.

**Proof.** The fact that $\sigma(H, n) \geq \sigma(\pi^*(H, n)) + 2$ follows from Theorem 3.1. Note that any $H$ with $\alpha(H) = 2$ and $d(H) = 2$ is a subgraph of $K_{k-2} + \overline{K}_2$ so that $\sigma(H, n) \leq \sigma(K_{k-2} + \overline{K}_2, n)$. Theorem 4.2 implies

$$\sigma(K_{k-2} + \overline{K}_2, n) = n(2k - 5) - k^2 + 4k - 1 + m = \sigma(\pi^*(H, n)) + 2.$$

The proposition follows. \hfill $\square$

Those graphs $H$ with $\alpha(H) = 2$ and $d(H) = 1$ have a considerably wider variety of structures. Any graph $H$ in this class is the complement of a triangle-free graph $G$ that is not a matching. The disjoint union of two cliques falls into this class, as does $K_k - tP_3$ and many other graphs of varying densities. We are able to verify Conjecture 1 for this diverse class of graphs. Our first observation is that any graph
Let \( H \) with \( \alpha(H) = 2 \) and \( d(H) = 1 \) must contain \( K_2 \cup K_1 \) as an induced subgraph, as this is the only graph on 3 vertices with maximum degree 1. This also immediately implies that \( m_1 = m_2 = 2 \). Therefore, if \( H \) is any graph of order \( k \) with \( \alpha(H) = 2 \) and \( d(H) = 1 \) and \( n \geq k \) is an integer, then (9) implies that
\[
\pi^*(H, n) = ((n - 1)^{k-3}, (k - 3)^{n-k+3}).
\]
The following lemma from [12] will be useful in the next proof.

**Lemma 4.4.** If \( \pi \) is a graphical sequence with a realization \( G \) containing \( H \) as a subgraph, then there is a realization \( G' \) of \( \pi \) containing \( H \) with the vertices of \( H \) having the \( |V(H)| \) largest degrees of \( \pi \).

We now show that Conjecture 1 holds when \( \alpha(H) = 2 \) and \( d(H) = 1 \).

**Proposition 4.5.** Let \( H \) be a graph of order \( k \) with \( \alpha(H) = 2 \) and \( d(H) = 1 \), and let \( n \) be a sufficiently large integer. Then
\[
\sigma(H, n) = \sigma(\pi^*(H, n)) + 2 = n(2k - 6) - k^2 + 5k - 4
\]

*Proof.* Let \( \pi \) be a nonincreasing, \( n \)-term graphic sequence with \( \sigma(\pi) \geq n(2k - 6) - k^2 + 5k - 4 \). Note that if \( n \) is sufficiently large, \( \sigma(\pi) \geq \sigma(K_{k-1}, n) \geq \sigma(K_{k-3} + \overline{K}_3, n) \).

We will show that \( \pi \) has a realization containing \( K_{k-3} + (K_2 \cup K_1) \) and, as we have previously observed that \( H \) must contain an induced copy of \( K_2 \cup K_1 \).

Let \( G \) be a realization of \( \pi \) that contains a copy of \( K_{k-3} + \overline{K}_3 \) on the \( k \) vertices of highest degree in \( G \). Such a realization exists by Lemma 4.4. Let \( S \) denote this subgraph, \( F \) denote the complete subgraph of order \( k - 3 \) and let \( I \) denote the independent set of order 3 in \( S \), so that \( S = F + I \). We can assume that \( F \) is comprised of the \( k - 3 \) vertices of highest degree in \( G \). If not, there are vertices \( x \) in \( I \) and \( y \) in \( F \) such that \( d(y) < d(x) \). We wish to create a realization of \( G \) containing a copy of \( K_{k-3} + \overline{K}_3 \) on the \( k \) vertices of highest degree such that \( x \) is in \( F \) and \( y \) is in \( I \). If \( x \) is adjacent to all the other vertices in \( S \), we can simply exchange the roles of \( x \) and \( y \). If \( x \) was not adjacent to exactly one vertex in \( I \), say \( v \), then as \( d(x) > d(y) \) there is some vertex \( w \) outside of \( S \) that is adjacent to \( x \) but not to \( y \). We will create a new realization of \( \pi \) by adding the edges \( yw \) and \( xv \) and deleting the edges \( yv \) and \( xw \). The case where \( x \) is not adjacent to exactly two vertices in \( I \) is handled similarly. Repeating this process allows us to create a realization of \( \pi \) containing \( K_{k-3} + \overline{K}_3 = F + I \) in which the \( k - 3 \) highest degree vertices of \( G \) lie in \( F \).

Let \( x_1 \) and \( x_2 \) be the vertices in \( I \) having the highest degrees, and note that \( \sigma(\pi) \geq \sigma(K_{k-1}, n) \) implies \( d(x_1) \) and \( d(x_2) \) are both at least \( k - 2 \). If there is any edge in the subgraph induced by \( I \), then \( G \) contains a copy of \( K_{k-3} + (K_2 \cup K_1) \) and we are done. Therefore, we may assume that \( I \) is an independent set. Let \( N_1 \) and \( N_2 \) denote \( N(x_1) \setminus S \) and \( N(x_2) \setminus S \), respectively, and note that both of these sets are nonempty since \( d(x_1) \) and \( d(x_2) \) are both at least \( k - 2 \). If \( y_1 \) and \( y_2 \) are distinct vertices in \( N_1 \) and \( N_2 \), respectively, then we may assume that \( y_1 \) and \( y_2 \) are adjacent. If they are not, then we would exchange the edges \( x_1y_1 \) and \( x_2y_2 \) for the nonedges \( x_1x_2 \) and \( y_1y_2 \), creating an edge in \( I \) and completing the proof.
The goal of the next part of this proof is to show that we may assume that there is some vertex $v$ in $F$ such that $d(v) \leq 4k$.

Consider first the case where $N_2 \subseteq N_1$ ($N_1 \subseteq N_2$ is handled identically) and let $w$ be a vertex in $N_2$. If $|N_1 \setminus N_2| > k$ then $d(w) > d(x_1)$ since $w$ is adjacent to every vertex in $N_1 \setminus N_2$. We therefore assume that $|N_1 \setminus N_2| \leq k$. Also note that $N_1 \cap N_2$ is a clique, and hence contains at most $k - 2$ vertices. There is some vertex $v$ in $F$ that is not adjacent to $w$, otherwise $d(w) > d(x_1)$, which contradicts our choice of $G$. Let $y$ be a neighbor of $v$ that does not lie in $S \cup N_1 \cup N_2$. If no such $y$ exists, then clearly $d(v) \leq 4k$. We claim that $wy$ is an edge of $G$, lest we could exchange the edges $x_1w, x_2w$ and $yw$ for the nonedges $wv, wy$ and $x_1x_2$ (see Figure 1), creating an edge in $I$. However, if the degree of $v$ is more than $4k$ there are at least $k - 1$ such choices for $y$. This implies that $d(w) \geq k + |N_1| > d(x_1)$, which contradicts our choice of $G$. Thus we may assume that $d(v) \leq 4k$.

Assume now that there is some vertex $w_1$ in $N_1 \setminus N_2$ and some vertex $w_2$ in $N_2 \setminus N_1$. We first show that $N_1 \cup N_2$ is complete. To accomplish this, we need only show that for any $w'_1$ in $N_1 \setminus N_2$, $w_1w'_1$ is an edge of $G$ (or symmetrically, if $w'_2$ is an element of $N_2 \setminus N_1$ then $w_2w'_2$ is an edge in $G$). If not, we can exchange the edges $x_1w_1, x_1w'_1$ and $x_2w_2$ for the nonedges $w_1w'_1, x_1w_2$ and $x_1x_2$, creating an edge in $I$ and completing the proof. Thus, since $N_1 \cup N_2$ is complete we may assume that $|N_1 \cap N_2| \leq k - 1$. Again, there is some $v$ in $F$ such that $w_2$ is not adjacent to $v$, lest $d(w_2) > d(x_2)$. Let $y$ be any neighbor of $v$ not in $S \cup N_1 \cup N_2$. Then $w_1$ is adjacent to $y$ or else we could exchange the edges $yw, x_1w_1$ and $x_2w_2$ for the nonedges $yw_1, vw_2$ and $x_1x_2$ (see Figure 2), creating an edge in $I$. If $d(v) > 3k$, then there are at least $k$ such choices for $y$, implying that $d(w_1) \geq k + |N_1 \cup N_2| - 1 > d(x_1)$, a contradiction.

Hence, we may assume that there is some vertex $v$ in $F$ such that $d(v) \leq 4k$. As a result, there are at most $(k - 4)(n - 1) + 4k$ edges adjacent to vertices in $F$, at most $12k$ edges adjacent to vertices in $I$ and, as both $N_1$ and $N_2$ have at most $4k$ vertices each, at most $4k(8k) = 32k^2$ edges adjacent to vertices in $N_1 \cup N_2$. This is at most $(k - 4)n + 32k^2 + 15k + 4$ edges. However, there are at least $\sigma(\pi)/2 = (k - 3 + o(1))n$ edges in $G$, so for $n$ sufficiently large there is some edge $yz$ in $G$ such that $y$ is not adjacent to any $w_1$ in $N_1$ and $z$ is not adjacent to any $w_2$ in $N_2$, where $w_1$ and $w_2$ may be the same vertex. We can therefore exchange the edges $x_1w_1, x_2w_2$ and
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Figure 2. $N_2 \not\subseteq N_1$ and $N_1 \not\subseteq N_2$

for the nonedges $w_1y, w_2z$ and $x_1x_2$, creating an edge in $I$, and completing the proof. □

Propositions 4.3 and 4.5 together imply Theorem 4.1. As mentioned above, there is quite a wide variety to the structures of those graphs $H$ having independence number 2, and yet we have demonstrated that $\sigma(H, n)$ for this class depends only on the value of $d(H)$, as suggested by Conjecture 1.

5. $H$-Saturated Graphs

Here we describe the relationship of $\sigma(H, n)$ to another extremal function $\text{sat}(n, H)$. We begin with the relevant terminology and results.

A graph $G$ is said to be $H$-saturated if $G$ contains no copy of $H$ as a subgraph and for any edge $e$ not in $G$, $G + e$ does contain a copy of $H$. The problem of determining the minimum number of edges in an $H$-saturated graph, denoted $\text{sat}(n, H)$, was first considered in 1963 by Erdős, Hajnal and Moon [6] for $H = K_t$. They determined that $\text{sat}(n, K_t) = (t - 2)(n - 1) - \binom{t - 2}{2}$, which arises from consideration of the split graph $K_{t-2} + \overline{K}_{n-t+2}$. The best known upper bound for an arbitrary graph $H$ is given by the following result of Kászonyi and Tuza [13].

**Theorem 5.1** ([13]). Let $u(H)$ be as defined above, and set

$$s(H) = \min\{e(H^*) | \alpha(H^*) = \alpha(H), \ |V(H^*)| = \alpha(H) + 1, H^* \subseteq H\}$$

then,

$$\text{sat}(n, H) \leq n(u(H) + \frac{s(H) - 1}{2}) - \frac{u(H)(u(H) + s(H))}{2}. \quad (16)$$

The reader should note that the bound given in Theorem 5.1 reflects the number of edges in the join of $K_{u(H)}$ and a graph which is (nearly) $(s - 1)$-regular. Comparing Theorem 5.1 to the construction of $\pi^*(H, n)$, we note that $d(H) \leq s(H)$ and hence that if $i \geq s(H)$, $n_i = 0$. Theorem 5.1 and Theorem 3.1 immediately imply the following result.
Theorem 5.2. Given a graph $H$, if there exists an $H' \subseteq H$ with $2u(H') + d(H') - 1 \geq 2u(H) + s(H) - 1$ then for $n$ sufficiently large we have
\[ 2\text{sat}(n, H) < \sigma(H, n). \] (17)

In particular, this result holds if $d(H) = s(H)$.

We strongly believe that the conclusion of Theorem 5.2 holds in general, even though the hypothesis does not. Therefore, we conjecture the following.

Conjecture 3. Let $H$ be a graph and let $n$ be a sufficiently large integer. Then
\[ 2\text{sat}(n, H) < \sigma(H, n). \]

As the problem of determining $\text{sat}(n, H)$ has proven difficult over time, we are not able to confirm Conjecture 3 in as many cases as Conjectures 1 and 2. We know that Conjecture 3 holds for complete graphs [6], [7], $tK_p$ and certain generalized friendship graphs [8], $C_4$ [12], [22], [24], and $K_{1,t}$ [13].

6. Conclusion

In light of Theorem 4.1, it may be interesting to individually consider classes of graphs with fixed independence number. This may be a fruitful direction, although the diversity in the structures of the $(\alpha(H) + 1)$ vertex induced subgraphs of such graphs rapidly increases. We feel that this line of investigation would move us closer to the goal of verifying either of Conjectures 1 and 2.

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REFERENCES

A General Lower Bound for Potentially H-Graphic Sequences

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